The Infinite Wisdom of Georg Cantor

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Throughout history, there are many examples of brilliant mathematicians who changed the world with new and astounding theories. Many challenged the common notions of their time, proposing new ways of conceiving shapes, numbers, and relationships. In his Elements, written around 300 BC, Euclid discarded the belief that visual representations could justify a hypothesis, and insisted on an axiomatic system of logical proofs. In 225 BC, Archimedes had the insight to realize that he could, in theory, construct a polygon whose area was arbitrarily close to that of a given circle. Over 1800 years later, this notion of arbitrarily or infinitely close was utilized by Leibniz and Newton in the creation of Calculus, a development that changed all sciences forever. Following this long line of innovators we come to a man named Cantor, whose theory of infinite sets was so groundbreaking that it has been dubbed "one of the most disturbingly original contributions to mathematics in 2500 years" (Burton, 625).

Georg Ferdinand Ludwig Phillip Cantor was born on March 3, 1845, in St. Petersburg, Russia. Many of his relatives were artists or musicians, thus from an early age Cantor was surrounded by an environment that fostered his incredible genius and creativity. Both of his parents may originally have been Jewish, but his father later converted to Protestantism and his mother to Catholicism (Maor, 54). This rich religious background led the young Cantor to develop a deep interest in theology, especially in questions concerning the nature of the infinite.

His father wanted him to study engineering, as this would be much more profitable than mathematics, so in 1862 Cantor started at a university in Zurich, Germany. After one semester he had had enough of this mundane science, and his father finally agreed to let him study mathematics. Cantor then transferred his studies to the University of Berlin, where he was fortunate enough to study under Weierstrass, Kummer, and (less fortunately) Kronecker. In 1867, at the age of only 22, he received his PhD for a thesis on number theory, and accepted an appointment at Halle University in 1869.

Intrigued by Weierstrass' groundbreaking and rigorous analysis, Cantor wrote a series of papers on representations of functions as trigonometric series. It was these papers that led him quite unexpectedly to the study of sets of points on the real line. This topic quickly revealed itself to be much more complex than it originally seemed, and Cantor decided to begin a complete investigation into the intricate workings of sets, both finite and infinite. His next paper, *Über eine Eigenshaft des Inbegriffes aller reellen algebraischen Zahlen* (*On a Property of the System of all the Real Algebraic Numbers*), marked the birth of Set Theory (Burton, 625).

It is important here to discuss the concept of the infinite up until Cantor's time. Mathematicians felt that there were two types of infinities, the *potentially infinite* and the *actual infinite*. Of these, only the former was acknowledged as something that could be used in mathematics. The potentially infinite referred to a process which could be repeated over and over, but which at any given step was still finite. The idea of the potentially infinite can be seen in the concepts of limits and mathematical induction. The actual infinite, on the other hand, was strictly forbidden. Even the legendary Gauss expressed this view in an 1831 letter to Schumacher: "As to your proof, I must protest most vehemently against your use of the infinite as something consummated, as this is never permitted in mathematics. The infinite is but a figure of speech" (quoted in Burton, 628). Cantor could not accept this idea. In his mind, there were clearly sets, or aggregates, which were infinite. This entirely new concept required the deep investigative powers of Cantor's genius, and we now turn our attention to some of his specific results.

We begin with a definition: "By an 'aggregate' we are to understand any collection into a whole M of definite and separate objects m of our intuition or our thought" (Cantor, 85)[1]. Of fundamental importance was the ability to compare the sizes of

[1] This definition, and several others in this paper, are given in Cantor's original (translated) wording. Although they seem quite esoteric, it is interesting to see the original work. Note that in the majority of the paper we use the modern terms "set" and "cardinality," instead of Cantor's "aggregate" and "power."

two sets. At first, it seems that we can simply count the elements of each set and compare these numbers. This works for finite sets, but what about those pesky infinites? Cantor had the burst of insight to realize the relative size of two sets could be discerned by establishing a correspondence between their elements. In his words, "We say that two aggregates M and N are 'equivalent,' in signs M~N or N~M, if it is possible to put them, by some law, in such a relation to one another that to every element of each one of them corresponds one and only one element of the other" (Cantor, 86). Today, we understand the "law" to be a function that is both one-to-one and onto, that is, a bijection. We then see that two sets M and N are equivalent if there exists a bijection f: $M \rightarrow N$ (note that, since f is a bijection and thus has an inverse, the order $N \rightarrow M$ or $M \rightarrow N$ is irrelevant). With a little consideration, we see that this notion of equivalence is quite sufficient; it makes perfect sense that two sets are equal in size if we can match up their elements.

Now we can actually use this definition to see what it means to be able to count something. We can easily count the elements of any finite set. For example, consider the set {a,b,c,d}. We see that there are four elements, but how did we arrive at this conclusion? Clearly, we can just start with the first element and count 1, 2, 3, 4. What we have actually done is exhibited the bijection $f:\{a,b,c,d\} \rightarrow \{1,2,3,4\}$ given by f(a)=1, f(b)=2, f(c)=3, and f(d)=4. Since it is the set $N=\{1,2,3,\ldots\}$ of natural numbers that we use to count, we can extend the concept of counting to infinite sets with the following definitions[2].

(1) A set S is said to be *denumerable* if there exists a bijection f: $N \rightarrow S$.
(2) A set S is said to be *countable* if it is either finite or denumerable.
(3) A set S is said to be *uncountable* if it is not countable.

Thus, the set N is fundamental to our understanding of infinite sets. We can also think of a set as being denumerable if we can list all of its elements, since the very act of creating the list involves placing one element 1st, one 2nd, and so on. With this in mind, Cantor set out to determine which sets were denumerable.

We immediately see that N is denumerable by the trivial bijection f: $N \rightarrow N$ given by f(n)=n. Significantly more interesting is the fact that the set E^+ of all positive even numbers is denumerable, as is shown by f(n)=2n. We have just proved that $E^+ \sim N$, but E^+ is clearly a proper subset of N! Is this a contradiction? It definitely goes directly against Euclid's fifth common notion from Elements: "The whole is greater than the part." Indeed, when Galileo discovered that "there are as many squares as there are numbers because they are just as numerous as their roots," he thought that something was seriously wrong (quoted in Burton, 627). Similar "paradoxes" were discovered by Bolzano in the early 1800s, but it was Cantor's close friend, the famous mathematician Richard Dedekind who, in 1888, not only accepted this sort of relationship, but formalized it with a definition: "A set M is infinite if it is equivalent to a proper subset of itself; in the contrary case, M is finite." This definition of an infinite set was quite a contrast to the common practice of saying that something was infinite if it was not finite. This concept, however revolutionary, is tame compared to many of the other remarkable results of Cantor's theory.

It is easy to show that the odd numbers, the squares, and many other subsets of N are also denumerable, but what about the set Z of all integers? This set should have at least twice as many elements as N, right? Not even close. We normally think of the integers as the set $\{\ldots,-2,-1,0,1,2,\ldots\}$, but there is nothing to stop us from listing them as $\{0,1,-1,2,-2,3,-3,\ldots\}$. What we have done here is listed the integers according to the bijection f: $N \rightarrow Z$ defined by $f(n) = \frac{n}{2}, \textit{for n even} \quad f(n) = \frac{1-n}{2}, \textit{for n odd}$. Thus, Z is denumerable. Finally, it can be proved that every subset of a countable set is countable, as the examples above illustrate. None of these results have been too surprising by today's standards; all of the sets considered involve only integers.

One might now wonder if the only denumerable sets are those consisting solely of integers. For example, we know the set Q of rational numbers is

[2] From here on we use N to be the set of natural numbers, which should not be confused with the general set N which Cantor used in his definitions.

dense[3], so how could we possibly list all of its elements? Remarkably, Cantor showed that this is indeed possible, but as Fermat would say the proof is too lengthy to include here. With similar proofs, he showed that NxN is denumerable, and even that a denumerable union of denumerable sets is denumerable.

Now it begins to seem that we have so many tricks up our sleeves that we can match up any two infinite sets. Surely if even the rationals are denumerable, we must wonder what is not. Cantor must also have wondered this, and his investigation into this matter brings us to a truly great theorem: The set R of real numbers is uncountable. Two proofs of this were given by Cantor, the most well known being his famous "diagonal proof." We wish to present the first proof, given in his 1874 publication, which requires a preliminary definition and theorem given in the appendix.

Theorem: The set R of real numbers is uncountable.

Proof: It suffices to show that the interval $I=(0,1)$ is uncountable, for if R were countable then the subset I would be countable by a previous result. If, to the contrary, we assume that $I=(0,1)$ is countable, we can list the elements of I as $I=\{x_1, x_2, x_3, \ldots\}$. First, select a closed subinterval I_1 of I such that $x_1 \notin I_1$. Next, select a closed subinterval I_2 of I_1 such that $x_2 \notin I_2$. Continuing in this fashion, we obtain a nested sequence of closed, bounded intervals $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq \ldots$ such that $x_n \notin I_n \ \forall n \in N$. Thus, the Nested Intervals Property guarantees the existence of a real number ξ such that $\xi \in I_n \ \forall n \in N$. But then $\xi \neq x_n \ \forall n \in N$. But $\xi \in I$ since $I \supseteq I_n \ \forall n \in N$, so we have produced an element of I that is not on our list, which contradicts that we had listed all the elements of I. Hence, (0,1), and therefore R, are uncountable. QED

An immediate consequence of this theorem is that the set R-Q of all irrational numbers is uncountable, for if R-Q were countable then $(R\text{-}Q) \cup (Q) = R$ would be countable. Consequently, there are infinitely many more irrationals than rationals.

We have now confirmed that uncountable sets exist, and thus that there are different "sizes" of infinity. Considering this, we would like some way of ordering these sizes, and for this we would like to have symbols representing each size. Cantor of course realized this, and gave the following definition: "We will call by the name 'power' or 'cardinal number' of M the general concept which, by means of our active faculty of thought, arises from the aggregate M when we make abstraction of the nature of its various elements m and of the order in which they are given" (Cantor, 86). Upon deciphering this into English, we realize that the cardinal number of a set is what we know about the set if we do not consider what its elements are: the only thing we can see in this case is its size. Thus, the cardinality of a set represents how large it is. The notation used for the cardinal number of a set S is $\overline{\overline{S}}$. A very important remark is that, for sets A and B, $\overline{\overline{A}} = \overline{\overline{B}} \Leftrightarrow A \sim B$. As a standard basis from which to work, Cantor defined $\overline{\overline{N}} = \aleph_0$ *and* $\overline{\overline{R}} = c$.[4] Full details on comparing the sizes of sets by ordering the cardinals are given in the appendix, but for now we will accept $\aleph_0 < c$.

We now must ask if there exist cardinals greater than c. That is, are there sets that are even bigger than R? Cantor was sure that there must be, and also thought that he knew exactly where to find one: the real plane. Specifically, he reasoned that there must be more points in the unit square than in the unit interval. After many fruitless attempts to prove this, in 1877 he finally ended up proving just the opposite: the unit square is equivalent to the unit interval! Thus, there are just as many points in the two dimensional plane as on the one dimensional line! This completely unexpected result prompted a letter from Cantor to Dedekind, exclaiming "I see it but I do not believe it" (quoted. in Dunham, 273). In fact, it was later proven that $\overline{\overline{R^n}} = c \ \forall n \in N$.

[3] A rational number is simply a fraction. To say that Q is dense means that between any two real numbers there lies a rational number.

[4] Considering the domain (0,1), the functions f(x)=(b-a)x + a and f(x)=tan(πx-π/2) show that, for any interval (a,b) of real numbers, we have (a,b)~(0,1)~R, and thus $\overline{\overline{(a,b)}} = \overline{\overline{(0,1)}} = \overline{\overline{R}} = c$.

Where, then, can we find a more abundant set than R? Still convinced that such sets must exist, Cantor persevered and was finally rewarded with the theorem that today bears his name.

Cantor's Theorem: If A is any set, then $\overline{\overline{A}} < \overline{\overline{P(A)}}$.[5]

Proof: We begin by showing that $P(A)$ is at least as large as A, and then show by contradiction that $P(A)$ is not equivalent to A.

First, consider f: $A \rightarrow P(A)$ given by f(a)={a}. This function maps all of the elements of A to some of the elements of $P(A)$ (it is 1-1), so $P(A)$ must be at least as large as A. Thus, $\overline{\overline{A}} \le \overline{\overline{P(A)}}$.

Now, assume $\overline{\overline{A}} = \overline{\overline{P(A)}}$. Then we have $A \sim P(A)$. By definition, this means there exists a bijection g: $A \rightarrow P(A)$. Now we consider the set $B=\{a \in A: a \notin g(a)\}$. B is clearly a subset of A, and so $B \in P(A)$. Hence, since g is a bijection, there must be some $x \in A$ such that $g(x)=B$. We now ask whether or not x is an element of B. There are two cases:

Case 1: Assume $x \in B$.

Then by definition of B, we have $x \notin g(x)$. But $g(x)=B$, so $x \notin B$, which contradicts our assumption. Thus, this case is impossible.

Case 2: Assume $x \notin B$.

Then by definition of B, we have $x \in g(x)$. But $g(x)=B$, so $x \in B$, which contradicts our assumption. Thus, this case is also impossible.

Since both cases lead to contradictions, we are forced to conclude that our initial assumption was false. That is, $\overline{\overline{A}} \ne \overline{\overline{P(A)}}$.

Hence, we have $\overline{\overline{A}} \le \overline{\overline{P(A)}}$ and $\overline{\overline{A}} \ne \overline{\overline{P(A)}}$. Therefore, $\overline{\overline{A}} < \overline{\overline{P(A)}}$.

QED

So: the set P(R) is larger than R, but this theorem actually goes far beyond finding a set with greater cardinality than R. Since, for any set A, P(A) is itself a set, we can construct P[P(A)], and P[P[P(A)]], and so on. We then obtain the following sequence: $\aleph_0 < c < \overline{\overline{P(R)}} < \overline{\overline{P(P(R))}} < \overline{\overline{P(P(P(R)))}} < \ldots$. Thus, we have a profound truth: there is an infinite hierarchy of infinities.

To us, Cantor's theory seems amazing. It is a beautiful example of mathematical ingenuity at its best, and "has been called the first truly original mathematics since the Greeks" (Dunham, 280). Unfortunately, in Cantor's time it was nothing short of blasphemy. Outraged at his use of the actual infinite, and even more at the transfinite arithmetic he created in his 1895 <u>Contributions to the Founding of the Theory of Transfinite Numbers</u>, the mathematical community erupted into one of the most bitter disputes in history. Most wounding to Cantor were the attacks of one of his former teachers, Leopold Kronecker.

Kronecker was undoubtedly a great mathematician, but also an ultraconservative one. He believed that any math not constructible from the natural numbers alone was ridiculous, and particularly disliked Weierstrass' Analysis. He once brought his esteemed colleague to tears with a comment concerning "the incorrectness of all these conclusions used in the so-called present method of analysis" (Burton, 631). If Kronecker could not handle ε's and δ's, one can imagine his feelings towards Cantor's infinities.

As a justification for his use of the actual infinite, Cantor once argued "to deny the actual infinite means to deny the existence of irrational numbers" (Maor, 55). Unfortunately, this is exactly what Kronecker did, apparently preferring the state of mathematics in the time of Pythagoras over that of the 19th century. Kronecker's extreme distaste for the new Set Theory seemed to devolve into a personal hatred of Cantor himself, and affected much of our genius' life. Cantor always wanted to get an appointment at a more prestigious university, but Kronecker, as a senior professor at the University of Berlin, made this impossible. He even went so far as to block many of Cantor's articles from being published in all but the smallest journals. It should be noted that Cantor did have some supporters at the time, most notably Weierstrass, Dedekind, and David Hilbert, but Kronecker's opposition was just too strong.

All of these personal attacks, along with the stress of his great mathematics (particularly his attempts to

[5] P(A) is the set of all subsets of A, and is called the power set of A.

prove the continuum hypothesis), had a devastating effect on Cantor[6]. In 1884 he had his first nervous breakdown, and was committed to a psychiatric clinic in Halle. After a short time he was released, and went on developing his theory. He was again in fine form in 1888, and after years of defending his work remarked "My theory stands as firm as a rock; every arrow directed against it will return quickly to its archer" (Dunham, 283). Tragically, in 1899 the death of his son Rudolph initiated another descent. Cantor was forced back to the psychiatric clinic in 1902, 1904, 1907, and 1911, and finally retired from Halle University in 1913. By this time his theories were beginning to gain much more widespread acceptance, but the damage had been done, and Cantor died in the clinic on January 6, 1918 (Dunham, 279).

This was indeed a tragic end to one of the most brilliant men the world has ever known, but his ideas live on, continuing to amaze and inspire countless mathematicians. Particularly, the great German mathematician David Hilbert would not allow Cantor's theory to die. He regarded this new field as "the finest product of mathematical genius and one of the supreme achievements of purely intellectual human activity" (quoted in Burton, 629). Hilbert also defended the Set Theory from future attackers, insisting "no one will expel us from the paradise that Cantor has created" (quoted in Dunham, 281). Clearly, Cantor left a great treasure for all future generations of mathematicians.

We conclude with a statement by Cantor himself, who never for a moment doubted the truth of his revolutionary mathematics:

> *This view, which I consider to be the sole correct one, is held by only a few. While possibly I am the very first in history to take this position so explicitly, with all of its logical consequences, I know for sure that I shall not be the last!* (quoted in Dunham, 280)

[6] Modern psychiatric analysis also shows that he may have been bipolar (Dunham, 279).

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Appendix

Preliminaries for Cantor's Proof that *R* is Uncountable

Definition: A sequence of intervals $I_n, n \in N$, is *nested* if the following chain of inclusions holds: $I_1 \supseteq I_2 \supseteq ... \supseteq I_n \supseteq I_{n+1} \supseteq ...$

Theorem (Nested Intervals Property): If $I_n, n \in N,$ is a nested sequence of closed bounded intervals, then there exists a number $\xi \in R$ such that $\xi \in I_n \ \forall n \in N.$

The Ordering of Cardinal Numbers

Definitions: Let *A* and *B* be sets.

(1) $\overline{\overline{A}} = \overline{\overline{B}}$ *iff* $A \sim B$; *otherwise* $\overline{\overline{A}} \neq \overline{\overline{B}}.$

(2) $\overline{\overline{A}} \leq \overline{\overline{B}}$ *iff there exists an injection* $(1-1)$ $f : A \to B.$

(3) $\overline{\overline{A}} < \overline{\overline{B}}$ *iff* $\overline{\overline{A}} \leq \overline{\overline{B}}$ *and* $\overline{\overline{A}} \neq \overline{\overline{B}}.$

Remarks:

(i) On the set of cardinal numbers, it can be shown that the above relation $\leq$ is reflexive, transitive, and antisymmetric, and that for any two elements x and y either $x \leq y$ or $y \leq x$. Thus, it is a total order on the set of cardinal numbers. If we accept the generalized continuum hypothesis, then $\leq$ is also a well ordering on the set of cardinal numbers.

(ii) Many of the familiar properties of $\leq$ on the real numbers, such as $x \leq y$ iff x<y or x=y, can be shown to carry over to the cardinal numbers.