Fractals: Psychedelically Mathematical

Jordan Porter

CSU Stanislaus

Introduction

Although originally discovered by mathematicians such as Gastone Julia in the early 19th century, fractals were not expanded upon until the advent of more powerful computers in the 1980s. Nowadays, they have widespread applications in the sciences. Furthermore, many types of fractals create truly beautiful images. However, many people believe that it is not only the images that make them beautiful, but the intricate mathematical properties behind them.

Mandelbrot and Julia Sets

- The two more widely known fractals are Mandelbrot and Julia sets. Although there is an infinite amount of Mandelbrot sets, the most famous set is the one generated by the function f(z) = z^2 + c, which is shown in Figure 1.
- Mandelbrot used geographic terms to describe regions of the Mandelbrot set. The massive figure dominating the image is called the “continent”. The thin protrusions from the continent are called the “tendrils”. The self-similar copies of the continent lying on the tendrils are referred to as the “islands”. Images of these regions are shown in Figures 2-4
- Orbits of points in the same 'bulb' of the Mandelbrot set all share the same period. In Figure 1, each bulb is colored according to the orbits of the points within it's bounds.

Julia Sets

- A Julia set can be generated for each point in a Mandelbrot set. For this reason, the two sets have many connections. Figure 5 shows a Julia set.
- The periods of the orbits of the points within a Julia set generated in a specific region of the accompanying Mandelbrot are the same as those points in that region. The color of Figure 5 shows the orbit of the points within. Note the color is the same as the region of the Mandelbrot set from which it was created.

The Mandelbrot Set

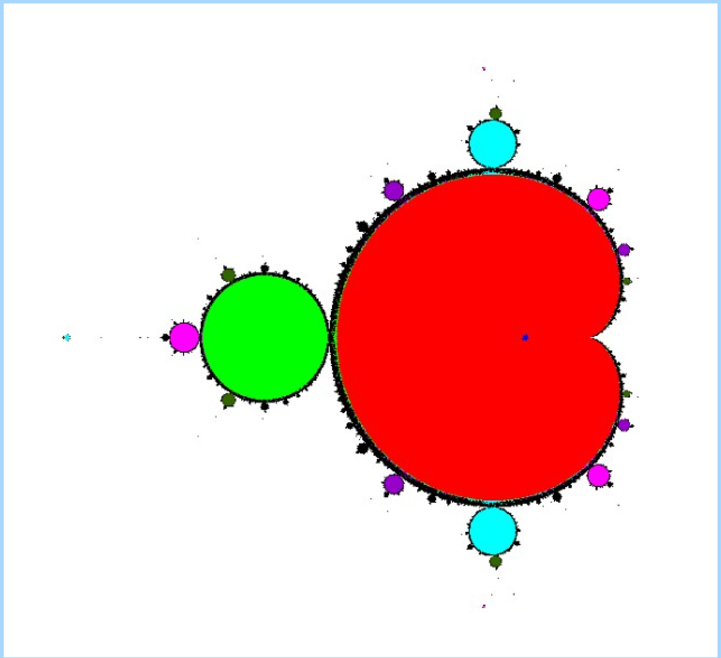
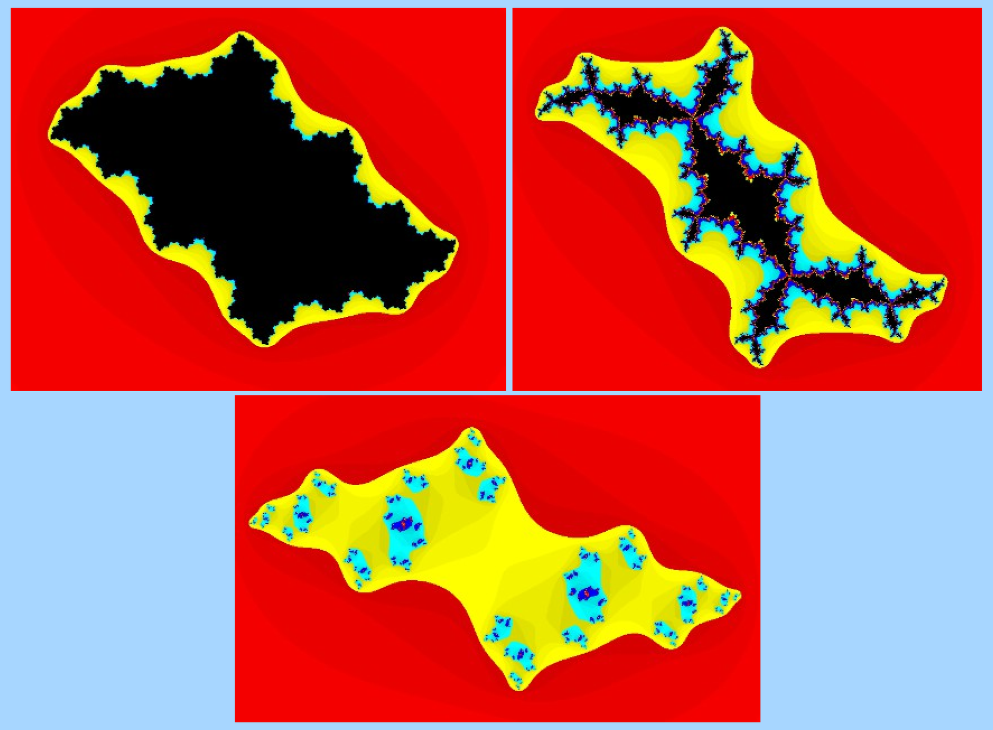


Figure 1: A classic Mandelbrot set, f(z) = z^2 + c. The colors show the orbits of points within the set. Areas in red have a period of one, green 2, teal 3, pink 4, etc.

Some Julia Sets



From top left to bottom middle: Figures 7, 8, and 9. These Julia sets were generated from different areas of the Mandelbrot set.

Mandelbrot Zooms

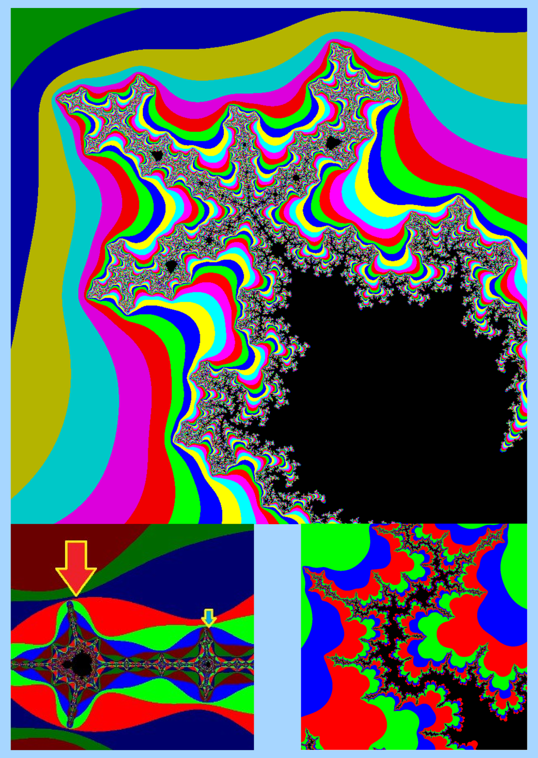


Figure 2 Shows a zoom on one of the Mandelbulbs. Notice the tendrils shooting off the edges. Figure 3 shows two islands off to the left of the main continent along the real axis. Figure 4 shows another tendril.

A Julia Set

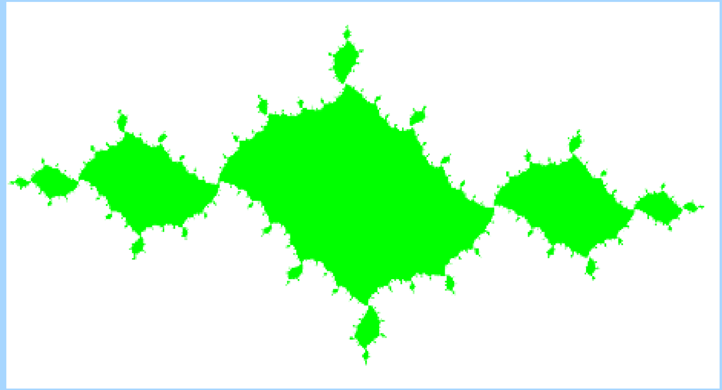
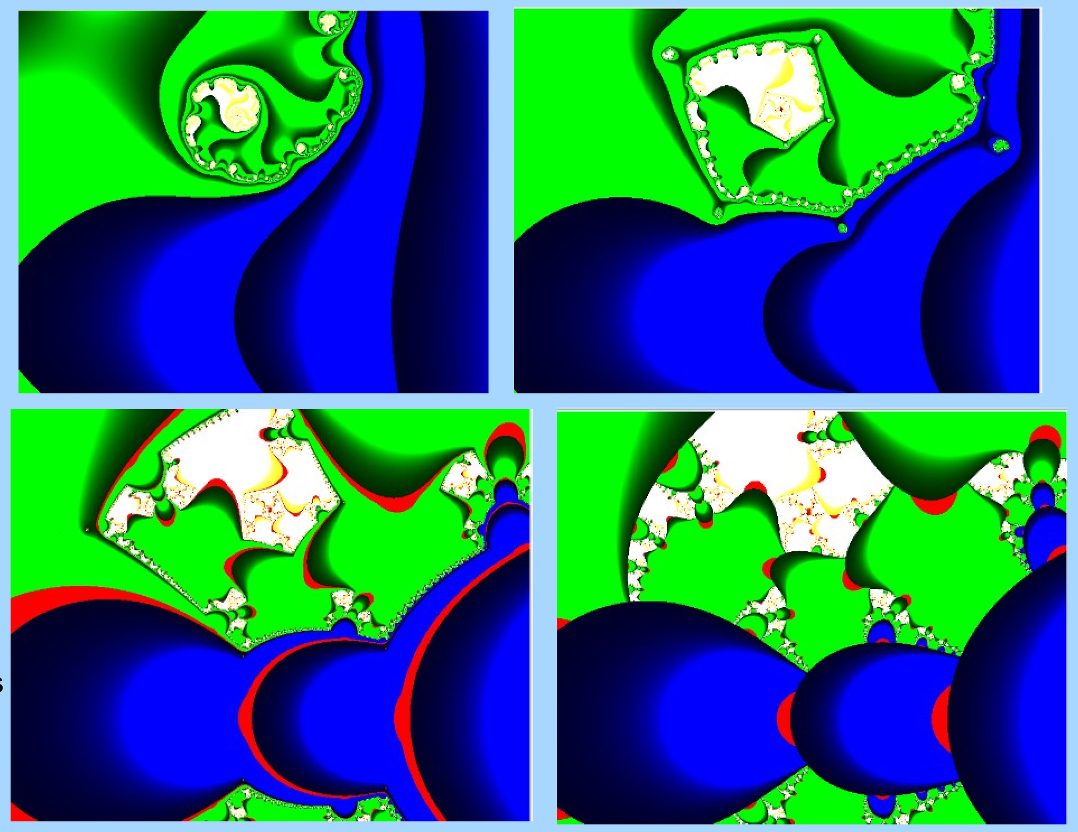


Figure 5: A Julia set with the equation f(z) = z^2 + c. This set is pulled from the green region of Figure 1. Notice how the period of the points within this set equal the period of the points in the basin from which it originated.

Maclaurin Series Approximations



From top left to bottom right: Figures 10-13. Figure 10 shows the Maclaurin series for the function f(z) = e^z with 5 terms, Figure 11 shows 10 terms, and Figure 12 shows 30 terms. Figure 13 shows the Julia set for the actual function .

Photograph of Gastone Julia

Julia Sets (cont.)

- The look of a Julia set is radically affected by the placement of it's value of c in it's associated Mandelbrot set. As shown in Figure 6.
- Sets generated from the interior of a Mandelbrot set are more structured and filled in, as in Figure 7.
- Sets generated near the boundary of a Mandelbrot set tend to be highly chaotic yet aesthetically pleasing, as shown in Figure 8.
- Sets generated outside the boundaries of it's associated Mandelbrot set look like hazy outlines of fractal images. These are referred to as “Fractal Dusts”. See Figure 9.

Series Visualizations

- Fractals can also be used to visualize the convergence of a Maclaurin series to the function that it is approximating. For instance, Figures 10-14 are sets created by the Maclaurin series expansion of e^z for n = 5, 10, and 30. Notice how the images slowly seem to converge to the actual fractal image for f(z) = e^z.

Selected References

James Ward Brown and Ruel V. Churchill. *Complex Variables and Applications*. New York: McGraw-Hill Education, 2014.

Devaney, Robert L. *Chaos, Fractals, and Dynamics*. Menlo Park: Addison-Wesley Publishing Company, 1990.

Falconer, Kenneth. *Fractal Geometry.* West Sussex: John Wiley and Sons Ltd, 2003.

Mandelbrot, Benoit B. *The Fractal Geometry of Nature.* W. H. Freeman and Co., 1982.