begin with $[0,1]$.
remove 1 of length $1/3$
remove 2 of length $1/9$
remove 4 of length $1/27$
remove 8 of length $1/81$
remove 16 of length $1/243$
and in the limit...
... is the Cantor set

Figure 1: Construction of the Cantor set $\partial \Omega_{CS}$.

Keep track of lengths $l_n$ and multiplicities $m_n$ at every stage $n$, but distinguish between the open and closed intervals.
For the Cantor set, the sequence of lengths $\mathcal{L}$ of the disjoint open intervals that make up its complement, called the Cantor string $\Omega_{CS}$, are:

$$\mathcal{L} = \left\{ l_n = 3^{-n} \mid \text{each length has multiplicity } m_n = 2^{n-1} \right\}.$$
Definition 1. The geometric zeta function of a fractal string $\Omega$ with lengths $\mathcal{L}$ is

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} \ell_j^s = \sum_{n=1}^{\infty} m_n l_n^s,$$

where $\text{Re } s > \dim_B(\partial \Omega)$.

Theorem 2. If a fractal string $\Omega$ (a bounded, open set) in $[0, 1]$ is of total length 1 and has an infinite number of lengths in its sequence $\mathcal{L}$, then

$$\dim_B(\partial \Omega) = \inf \left\{ \sigma \in \mathbb{R} \mid \sum_{n=1}^{\infty} m_n l_n^\sigma < \infty \right\},$$

where $\partial \Omega = [0, 1] \setminus \Omega$. 
The Cantor set $\partial \Omega_{CS}$ has dimension

$$D = \text{dim}_B(\partial \Omega) = \log_3 2.$$ 

How’s that?

$$\zeta_C(s) = \sum_{n=1}^{\infty} m_n l_n^s$$

$$= \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^{ns}}$$

$$= \sum_{n=1}^{\infty} 2^{n-1} \cdot 3^{-ns}$$

$$= \frac{3^{-s}}{1 - 2 \cdot 3^{-s}}$$

Thus, when is the denominator of the fraction equal to 0?

$$1 - 2 \cdot 3^{-s} = 0 \Rightarrow$$

$$\frac{1}{2} = 3^{-s} \Rightarrow$$

$$-s = \log_3 \frac{1}{2} \Rightarrow$$

$$s = \log_3 2.$$
Measures!

A measure is like a function on sets, but with sensible restrictions. For instance, let $V \subset U$ and $U_n$ for $n \in \mathbb{N}$ be some intervals in $[0, 1]$ and let $\beta$ be a (positive) measure on $[0, 1]$. Then,

$$\beta(\cdot) : \{\text{intervals in } [0,1]\} \rightarrow \mathbb{R}_0^+ \cup \{\infty\},$$

$$\beta(\emptyset) = 0,$$

$$\beta(U) \geq 0,$$

$$\beta(U) \geq \beta(V),$$

and under certain other conditions (such as having the $U_n$ disjoint),

$$\beta\left(\bigcup_{n=1}^\infty U_n\right) = \sum_{n=1}^\infty \beta(U_n).$$
Figure 4: The first few stages in the construction of a mass distribution $\beta$ on the Cantor set. At each stage, mass is split from the previous stage in ratios of $1/3$ on the left and $2/3$ on the right.

How should we collect the intervals in a meaningful way in order to generate? What did you come up with? Remember that binomial coefficients yield the multiplicities of the masses at every step. They are:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

where $k$ is the number of powers of 2 in the weight, and $n$ is the stage.
Consider the exponent $\alpha$ is the exponent that satisfies $|U|^{\alpha} = \mu(U)$. Equivalently, we have the following definition:

**Definition 3.** The regularity $\alpha(U)$ of a (Borel) measure $\mu$ on a subset $U \subset [0,1]$ with range in $[0,\infty]$ is

$$\alpha(U) = \frac{\log \mu(U)}{\log |U|},$$

where $|\cdot| = \lambda(\cdot)$ is the Lebesgue measure on $[0,1]$. 

Figure 5: The solid black blocks correspond to the closed intervals with regularity $\alpha(1,2)$. 
Figure 6: The solid black blocks correspond to the closed intervals with regularity $1/2$.

For the intervals $U$ in the family $\mathcal{P}$, the possible mass values are $\beta(U) = \frac{2^k}{3^n}$ with length $|U| = \frac{1}{3^n}$. So, the regularity values are

$$\alpha(U) = \log \frac{2^k}{3^n}.$$

To exploit the properties of logarithms and in order to come up with families of sequences of lengths, let $k_1$ be the number of powers of 2 that appear in the mass $\beta(U)$. Let $k_2$ be the number of stages it took to get the $k_1$ powers of 2. Then, let $n$ be the index of the clump of stages with $k_2$ steps. What do we get?
Figure 7: The solid black blocks correspond to the closed intervals with regularity $\alpha(1, 2)$.

For the measure $\sigma$ and the family $\mathcal{P}$ of partitions given by the open and closed intervals in the construction of the Cantor set, the regularity values are

$$\alpha = \alpha(k_1, k_2) = \frac{\log (2^{nk_1}/3^{nk_2})}{\log (1/3^{nk_2})} = 1 - \frac{k_1}{k_2} \log_3 2,$$

where $k_1$ and $k_2$ are relatively prime non-negative integers such that $k_1 < k_2$. Thus,

$$\zeta_\mathcal{P}^\beta(\alpha(k_1, k_2), s) = \sum_{n=1}^{\infty} \left(\frac{nk_2}{nk_1}\right)^{3^{-nk_2s}},$$

for $\text{Re } s$ large enough.
Here’s the big idea: What are the dimensions associated with the measure $\beta$? For each regularity $\alpha(k_1, k_2)$ we get a partition zeta function, and these functions have unique real numbers at which they diverge. Then, as with geometric zeta functions, these values yield a dimension associated with the given regularity.

$$
\zeta_\beta^\beta(\alpha(k_1, k_2), s) = \sum_{n=1}^{\infty} \left( \frac{n k_2}{n k_1} \right) 3^{-n k_2 s},
$$

So, determine where these functions diverge. Piece of cake!
The measure $\beta$ is multifractal. In our setting, this means that the distribution of weight generates a spectrum of dimensions. The abscissa of convergence function $f$ for the measure $\beta$ has the form

$$f(\alpha) = \frac{(\alpha - 1)}{\log_3 2} \cdot \log_3 \left( \frac{-(\alpha - 1)}{\log_3 2} \right) - \left(1 + \frac{(\alpha - 1)}{\log_3 2}\right) \cdot \log_3 \left(1 + \frac{(\alpha - 1)}{\log_3 2}\right).$$
Figure 9: Graph of the abscissa of convergence function $f$ for the measure $\beta$.

The maximum of $f$ is attained at $\alpha = \alpha(1, 2) = 1 - (1/2) \log_3 2$ and this value coincides with the box dimension of the support of the measure $\beta$. That is,

$$\dim_B(supp(\beta)) = \max\{ f(\alpha) \mid \alpha = \alpha(k_1, k_2) \} = \log_3 2.$$
Homework. Really.

1. Prove that the total length (or initial length) does not affect the box dimension of a fractal string with sequence of lengths $\mathcal{L} = \{\ell_j\}_{j=1}^\infty$.

2. Prove that for the measure $\beta$, the maximum value of $f$ is attained at $\alpha = \alpha(1, 2) = 1 - (1/2) \log_3 2$. BIG hint: This was done in my thesis. You can get a copy of my thesis from my website:

   http://www.csustan.edu/math/rock

Click on the familiar picture (not my face). See pages 45-60.

References


- Website for Mandelbrot set applet:
  http://math.hws.edu/xJava/MB/