Fractals and Fractal Dimensions

An Alternative Method for Computing Box Dimension

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A typical notion of dimension: Line segments are 1D, squares are 2D, cubes are 3D...
The dimensions $D = 1$ and $2$ for each type of figure above satisfy the following equivalent equations:

$$N_\varepsilon = \varepsilon^{-D} \iff D = \frac{\log N_\varepsilon}{-\log \varepsilon}$$

where $\varepsilon$ is the size of the side length of the boxes used to cover the objects and $N_\varepsilon$ is the minimum number of boxes required to cover them.
The method of counting boxes allows one to define a notion of dimension that assigns a real value (as opposed to simply an integer value) to a subset of Euclidean space.

**Definition 1.** The box dimension $\dim_B$ of a bounded subset $F$ of $\mathbb{R}^m$ is given by the following limit (when it exists):

$$\dim_B(F) = \lim_{\varepsilon \to 0^+} \frac{\log N_\varepsilon(F)}{-\log \varepsilon},$$

where $N_\varepsilon(F)$ is the smallest number of “cubes” with side length $\varepsilon$ that cover $F$. 
begin with [0,1] remove 1 of length 1/3
remove 2 of length 1/9
remove 4 of length 1/27
remove 8 of length 1/81
remove 16 of length 1/243
and in the limit...
... is the Cantor set

Figure 1: Construction of the Cantor set.

Note that when boxes of size $\varepsilon = 3^{-n} = 1/3^n$ are used, a minimum number $N_\varepsilon(F) = 2^n$ boxes are required to cover the set. Thus,

$$D = \lim_{\varepsilon \to 0^+} \frac{\log N_\varepsilon(F)}{-\log \varepsilon} = \lim_{n \to \infty} \frac{\log (2^n)}{-\log (3^{-n})} = \log_3 2 \approx 0.6309$$
Figure 2: The Cantor string $\Omega_{CS}$: the complement of the Cantor set.

There is an alternate method for computing the box dimension (a.k.a. Minkoswki dimension) of sets that are similar to the Cantor set. This alternative requires consideration of the complements of such sets, in particular the sequence of lengths associated with this complement.

In the case of the Cantor set, this amounts to keeping track of the lengths of the intervals that were removed in the construction of the Cantor set, along with their multiplicities.

Side note: Box dimension is not always equal to the Hausdorff dimension, which is perhaps more well-known and widely used. However, box dimension is the right notion for our purposes.
Figure 3: The lengths of the Cantor string.

For the Cantor set, the sequence of lengths $\mathcal{L}$ of the disjoint open intervals that make up its complement, called the *Cantor string* $\Omega_{CS}$, are:

$$\mathcal{L} = \left\{ 3^{-n} \mid \text{each length has multiplicity } 2^{n-1} \right\}.$$
Recall that the box dimension of the Cantor set is \( \log_3 2 \).

Consider the following series, where \( s \in \mathbb{R} \):

\[
\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^{ns}} = \sum_{n=1}^{\infty} 2^{n-1} \cdot 3^{-ns} = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}}
\]

Look at that last line: Where does this closed form become undefined?
\[
\frac{3^{-s}}{1 - 2 \cdot 3^{-s}}
\]

That is, when is the denominator of the fraction (the closed form of the series) equal to 0?

\[
1 - 2 \cdot 3^{-s} = 0 \implies \frac{1}{2} = 3^{-s} \implies -s = \log_3 \frac{1}{2} \implies s = \log_3 2.
\]

Look familiar? This is not a coincidence. In fact, when the lengths of a fractal string are the terms (when taken to the \(s\) power) of a series, the \textit{abscissa of convergence} is the dimension of the complementary fractal.
Remember, we are interested in sets that are complements in the unit interval $[0, 1]$ of fractals like the Cantor set.

**Definition 2.** A fractal string $\Omega$ is a bounded open subset of the real line. The collection of lengths $\ell_j$ of the disjoint intervals is denoted $\mathcal{L}$. 
The following theorem summarizes this alternative method for computing the box dimension using fractal strings:

**Theorem 3.** If a fractal string $\Omega$ in $[0,1]$ is of total length 1 and has an infinite number of lengths in its sequence $L$, then

$$\dim_B(\partial \Omega) = \inf \left\{ \sigma \in \mathbb{R} \mid \sum_{j=1}^{\infty} \ell_j^\sigma < \infty \right\},$$

where $\partial \Omega = [0,1] \setminus \Omega$.

In order to understand the proof of this theorem, so more tools are needed. Recall the definition of the box (Minkowski) dimension of a set $F$:

$$\dim_B(F) = \lim_{\varepsilon \to 0^+} \frac{\log N_\varepsilon(F)}{-\log \varepsilon}. $$
Figure 4: Approximation of the Cantor string $\Omega_{CS}$ and the volume $V(\varepsilon)$ for $1/54 < \varepsilon < 1/18$.

The one-sided volume of the tubular neighborhood of radius $\varepsilon$ of $\partial \Omega$ is

$$V(\varepsilon) = \lambda(\{x \in \Omega \mid \text{dist}(x, \partial \Omega) < \varepsilon\})$$

$$= \sum_{j: \ell_j \geq 2\varepsilon} 2\varepsilon + \sum_{j: \ell_j < 2\varepsilon} \ell_j,$$

where $\lambda(\cdot) = |\cdot|$ denotes the Lebesgue measure.

For the boundary $\partial \Omega$ of a fractal string $\Omega$ in $[0, 1]$ with total length 1, there is an alternative definition of the box (Minkowski) dimension:

$$\dim_B(\partial \Omega) = D_L := \inf\{\alpha \geq 0 \mid \limsup_{\varepsilon \to 0^+} V(\varepsilon)^{\alpha - 1} < \infty\}.$$

Note that one may refer directly to the box (Minkowski) dimension of the sequence of lengths $L$. Actually, this notion of simply considering lengths is really important, so try to keep it in mind.
Let’s look at the proof of Theorem 3.

Proof. Let \( d > D \). With the alternative definition of \( \dim_{\mathcal{M}} \), \( \exists \) constant \( C_1 \) such that \( V(\varepsilon) \leq C_1 \varepsilon^{1-d} \). Choosing \( \varepsilon = \ell_n/2 \) yields

\[
n\ell_n \leq n\ell_n + \sum_{j=n+1}^{\infty} \ell_j = V(\ell_n/2) \leq C_1(\ell_n/2)^{1-d}.
\]

Thus, \( \forall s > 0, \ell_s \leq C_2 n^{-s/d} \), for some \( C_2 > 0 \). Therefore, the \( \zeta_L(s) \) converges for \( s > d \), so \( \sigma \leq d \), which holds \( \forall d > D \). We obtain \( \sigma \leq D \).

(Continued...)
To show $\sigma \geq D$, let $\sigma < s < 1$ ($\sigma = 1 \Rightarrow D = \sigma$ since the volume is bounded). Thus, $\zeta_L(s)$ converges. The sequence of lengths is nonincreasing, so

$$n\ell_n^s \leq \sum_{j=1}^{n} \ell_j^s \leq \zeta_L(s).$$

Thus, $\ell_n \leq (C_3/n)^{1/s}$, $\forall n \geq 1$ and some $C_3 > 0$. Given $\varepsilon > 0$, it follows that $\ell_n < 2\varepsilon$ for $n > C_3(2\varepsilon)^{-s}$. We now split $V(\varepsilon)$ using two estimates: For $j \leq C_3(2\varepsilon)^{-s}$, estimate $j$-th term by $2\varepsilon$, and for $j > C_3(2\varepsilon)^{-s}$, use $(C_3/j)^{1/s}$. Therefore, for some $C_4 > 0$,

$$V(\varepsilon) \leq C_4(2\varepsilon)^{1-s}.$$

It follows that $D \leq s$, which holds $\forall s > \sigma$. So $D \leq \sigma$ and, therefore, $D = \sigma$. $\square$
The following function has many more properties than will be explored today.

**Definition 4.** The geometric zeta function of a fractal string $\Omega$ with lengths $\mathcal{L}$ is

$$
\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} l_j^s = \sum_{n=1}^{\infty} m_n l_n^s,
$$

where $\Re(s) > \dim_B(\partial \Omega)$.

Next time, we will look at a generalization of the geometric zeta function that works on fractal measures (as opposed to fractal sets). Keep in mind that the underlying fractal does not contribute to the computation of the geometric zeta function and the box (Minkowski) dimension, but the lengths of the corresponding fractal string (if there is one) do.
Definition 5. The set of complex dimensions of a fractal string $\Omega$ with lengths $\mathcal{L}$ is

$$D_{\mathcal{L}}(W) = \{\omega \in W \mid \zeta_{\mathcal{L}} \text{ has a pole at } \omega\},$$

where $W$ is a certain region in the complex plane.
Nifty things one can do with fractals strings, zeta functions, and complex dimensions:

- Find the box-counting dimension of the complements of fractal strings (Theorem 3).
- Find the volume of the inner $\epsilon$-neighborhood of the boundary of certain fractals.
- Investigate properties of fractal strings and multifractal measures.
- Show, in a new way, that the Critical Zeros of the Riemann Zeta Function do not lie in vertical arithmetic progression.
- Reformulate the Riemann Hypothesis as an inverse spectral problem.

For the Cantor String, the geometric zeta function is

$$
\zeta_L(s) = \zeta_{CS}(s) = \sum_{n=1}^{\infty} 2^{n-1} 3^{-ns} = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}}.
$$

Upon meromorphic continuation, we see that the last equation above holds for all $s \in \mathbb{C}$, hence

$$
\mathcal{D}_L = \mathcal{D}_{CS} = \left\{ \log_3 2 + \frac{2im\pi}{\log 3} \mid m \in \mathbb{Z} \right\}.
$$
References


• Website for Mandelbrot set applet:
  http://math.hws.edu/xJava/MB/