Fractals and Fractal Dimensions

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The Mandelbrot Set

$$M = \{c \in \mathbb{C} \mid 0 \to c \to c^2 + c \to \ldots \text{ remains bounded}\}$$

This set is fully described by iterating the complex-valued function

$$f(z) = z^2 + c$$

where c is a fixed complex number. How? Ask me later.

Visit my website: `www.csustan.edu/math/rock`

and click on the "Mandelbrot Set applet" link.

This will take you takes you to: `http://math.hws.edu/xJava/MBold/index`

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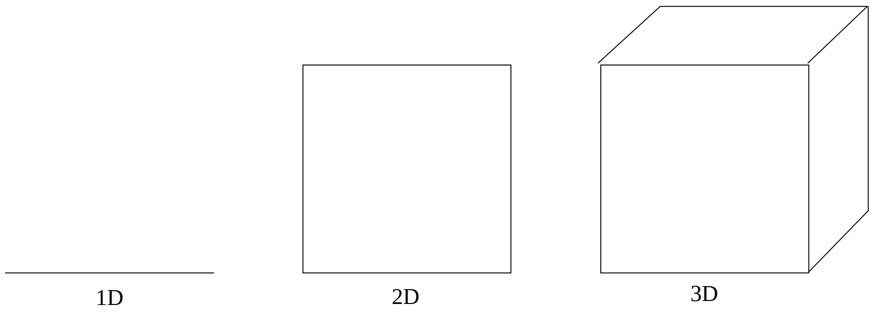
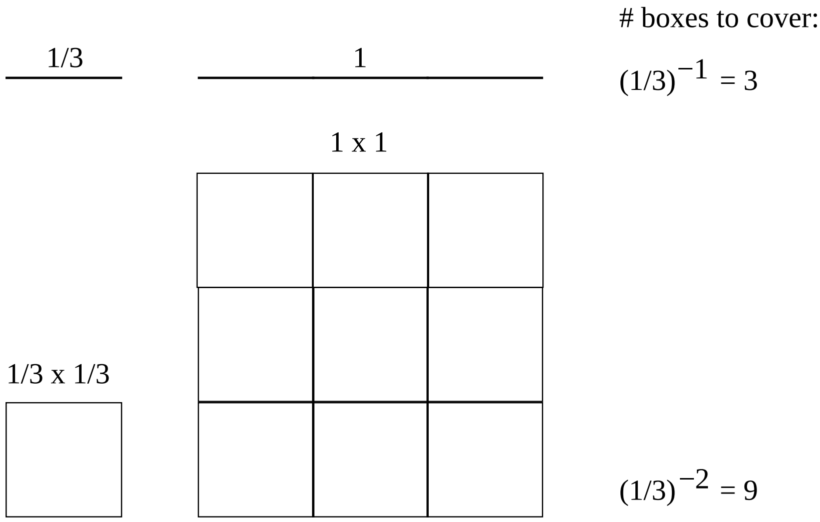
DIMENSION



A typical notion of dimension: Line segments are 1D, squares are 2D, cubes are 3D…



The dimensions $D = 1$ and 2 for each type of figure above satisfy the following equivalent equations:

$$N_\varepsilon = \varepsilon^{-D} \Leftrightarrow D = \frac{\log N_\varepsilon}{-\log \varepsilon}$$

where ε is the size of the side length of the boxes used to cover the objects and N_ε is the minimum number of boxes required to cover them.

The method of counting boxes allows one to define a notion of dimension that assigns a real value (as opposed to simply an integer value) to a subset of Euclidean space.

Definition 1. *The box dimension* $\dim_B$ *of a bounded subset* F *of* $\mathbb{R}^m$ *is given by the following limit (when it exists):*

$$\dim_B(F) = \lim_{\varepsilon \to 0^+} \frac{\log N_\varepsilon(F)}{-\log \varepsilon},$$

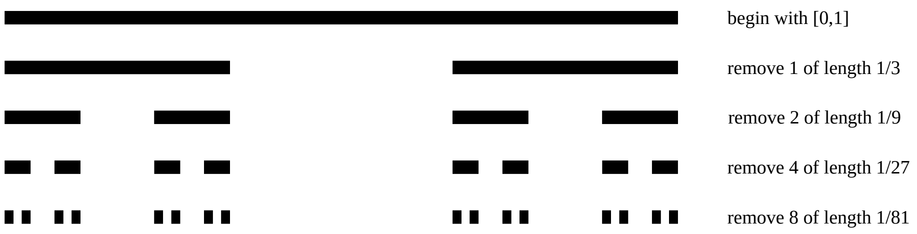
where $N_\varepsilon(F)$ *is the smallest number of "cubes" with side length* ε *that cover* F*.*

Step 0 in the construction of the Cantor set.

Step 1. Note that there are now 2 intervals of length $1/3$.

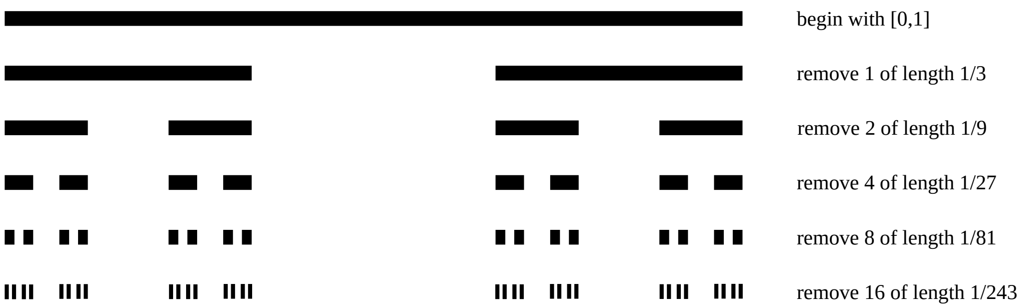
Step 2 yields 4 intervals of length 1/9.

Step 3 yields 8 intervals of lengths $1/27$. Do you see the pattern?

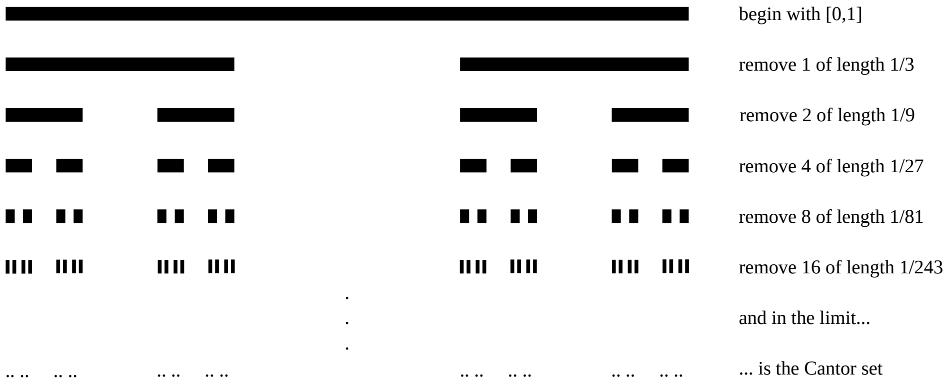


Step 4. The pattern is as follows:

For Step n, there are 2^n intervals of size $1/3^n = 3^{-n}$.

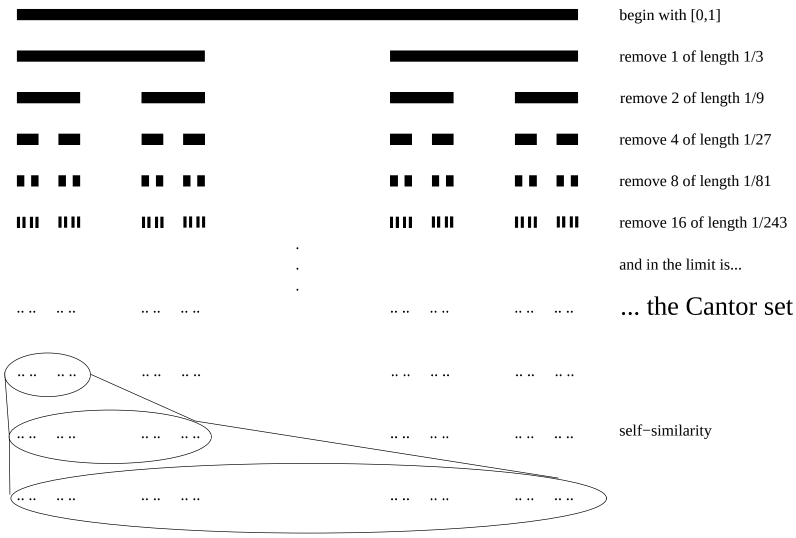


Step 5 yields $2^5 = 32$ intervals of length $3^{-5} = 1/243$.

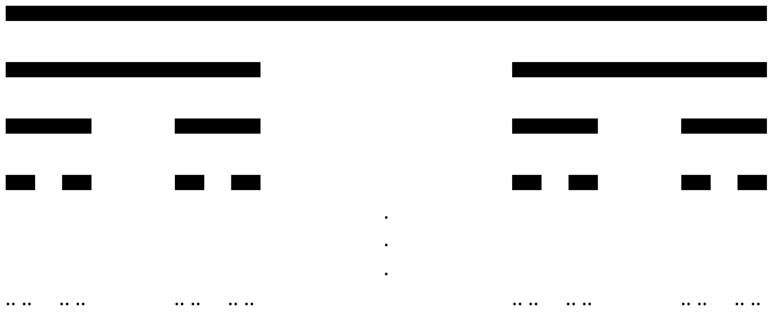


An approximation of the Cantor set, which is what you would get in the limit.

Note, to compute the dimension, we need to know how many intervals ($N = 2^n$) of whatever length ($\varepsilon = 3^{-n}$) are required to cover the set.

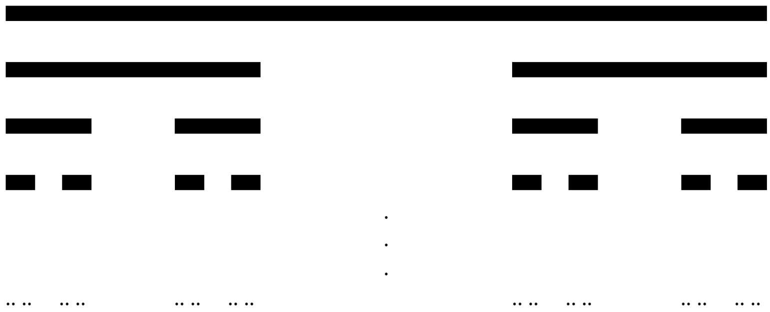


The Cantor set exhibits a property called "self-similarity".



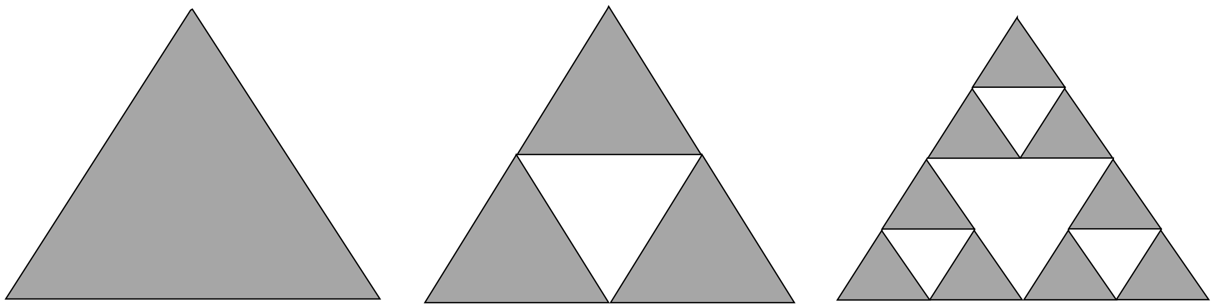
The Cantor set exhibits the following properties:

- It has length 0.
- It is a closed and perfect subset of $[0, 1]$ (every point is a limit point).
- It is uncountable.
- It is self-similar.
- It has box dimension equal to $\log_3 2$. (?)

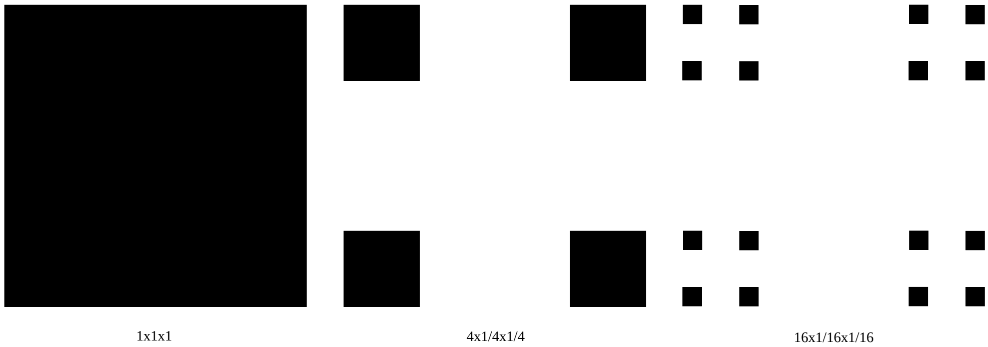


Note that when boxes of size $\varepsilon = 3^{-n} = 1/3^n$ are used, a minimum number $N_\varepsilon(F) = 2^n$ boxes are required to cover the set. Thus,

$$\lim_{\varepsilon\to 0^+} \frac{\log N_\varepsilon(F)}{-\log \varepsilon} = \lim_{n\to\infty} \frac{\log\left(2^n\right)}{-\log\left(3^{-n}\right)} = \log_3 2 \approx 0.6309$$



Can you determine the box dimension of the set you get in the limit? Try it! Assume that the length of the sides at Step 0 is 1. The set you get in the limit is called the Sierpinski Gasket.



Can you determine the box dimension of the set you'd get in the limit? Try it! What does this say about objects with integer dimension?

Find the box-counting dimension for each of the following objects. Google the names and see if you can find a nice description of the constructions of the fractals in 6 to 10. If you like, use a calculator to see if the numbers are between or equal to 0,1,2, or 3.

1. The line segment with length 1 (use $s = 1/3$).

2. The line segment with length 1 (use $s = 1/10$).

3. The square with side length 1 (use $\varepsilon = 1/3$).

4. The Cantor set that has the 2nd and 4th intervals of length 1/5 removed at **Step 1**.

5. The Cantor set that has just the middle interval of length 1/11 removed at **Step 1**.

6. The Koch Snowflake.

7. The Quadratic Koch Island.

8. The Sierpinski Carpet.

9. The Sierpinski Ternary Gasket.

10. The Menger Sponge.

Hint for problems 4–10: Draw Steps 0 and 1 to help see (literally) what ε and N should be.

Definition 2. *A* <u>*fractal string*</u> Ω *is a bounded open subset of the real line. The collection of lengths* ℓ_j *of the disjoint intervals is denoted* $\mathcal{L}$.

Theorem 3. *If a fractal string* Ω *in* $[0,1]$ *is of total length 1 and has an infinite number of lengths in its sequence* $\mathcal{L}$*, then*

$$\dim_B(\partial\Omega) = \inf\left\{\sigma \in \mathbb{R} \mid \sum_{j=1}^{\infty} \ell_j^{\sigma} < \infty\right\},$$

where $\partial\Omega = [0,1] \setminus \Omega$.

Definition 4. *The* <u>*geometric zeta function*</u> *of a fractal string* Ω *with lengths* $\mathcal{L}$ *is*

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} \ell_j^s = \sum_{n=1}^{\infty} m_n l_n^s,$$

where $\mathrm{Re} s > \dim_B(\partial\Omega)$.

Definition 5. *The set of* <u>*complex dimensions*</u> *of a fractal string* Ω *with lengths* $\mathcal{L}$ *is*

$$\mathcal{D}_{\mathcal{L}}(W) = \{\omega \in W \mid \zeta_{\mathcal{L}} \text{ has a pole at } \omega\}.$$

where W *is a certain region in the complex plane.*

Nifty things one can do with fractals strings, zeta functions, and complex dimensions:

- Find the box-counting dimension of the complements of fractal strings (Theorem 3).
- Find the volume of the inner ϵ-neighborhood of the boundary of certain fractals.
- Investigate properties of fractal strings and multifractal measures.
- Show, in a new way, that the Critical Zeros of the Riemann Zeta Function do not lie in vertical arithmetic progression.
- Reformulate the Riemann Hypothesis as an inverse spectral problem.

For the Cantor String, the geometric zeta function is

$$\zeta_{\mathcal{L}}(s) = \zeta_{CS}(s) = \sum_{n=1}^{\infty} 2^{n-1}3^{-ns} = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}}.$$

Upon meromorphic continuation, we see that the last equation above holds for all $s \in \mathbb{C}$, hence

$$\mathcal{D}_{\mathcal{L}} = \mathcal{D}_{CS} = \left\{ \log_3 2 + \frac{2im\pi}{\log 3} \mid m \in \mathbb{Z} \right\}.$$

The complex dimensions of the Cantor String.

The complex dimensions of another fractal string.

The first few stages in the construction of a mass distribution ν on the Cantor set. At each stage, mass is split from the previous stage in ratios of 1/3 on the left and 2/3 on the right.

Definition 6. *The* <u>*regularity*</u> $A(U)$ *of a (Borel) measure* μ *on a subset* $U \subset [0,1]$ *with range in* $[0,\infty]$ *is*

$$A(U) = \frac{\log \mu(U)}{\log |U|},$$

where $|\cdot| = \lambda(\cdot)$ *is the Lebesgue measure on* $[0,1]$.

Equivalently, $A(U)$ is the exponent α that satisfies

$$|U|^\alpha = \mu(U).$$

Definition 7. *For a measure* μ *and ordered family of partitions* $\mathfrak{P}$*, the* <u>*partition zeta function*</u> *with regularity* α *is*

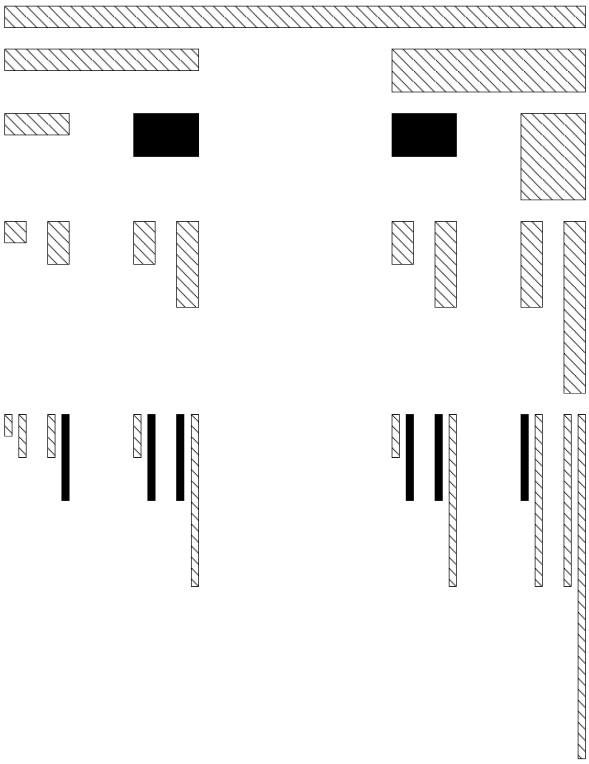
$$\zeta^\mu_{\mathfrak{P}}(\alpha, s) = \sum_{n=1}^{\infty} \sum_{k=1}^{p_n} \delta_\alpha(P_n^k)|P_n^k|^s,$$

where $\delta_\alpha(P)$ *equals 1 if* $A(P) = \alpha$ *and equals 0 otherwise,* p_n *is the number of intervals in the partition* $\mathcal{P}_n$*, and* $\mathrm{Re} s$ *is large enough.*

For the measure ν and the family $\mathfrak{P}$ of partitions given by the open and closed intervals in the construction of the Cantor set, the regularity values are

$$\alpha = \alpha(k_1, k_2) = \frac{\log(2^{nk_1}/3^{nk_2})}{\log(1/3^{nk_2})} = 1 - \frac{k_1}{k_2}\log_3 2,$$

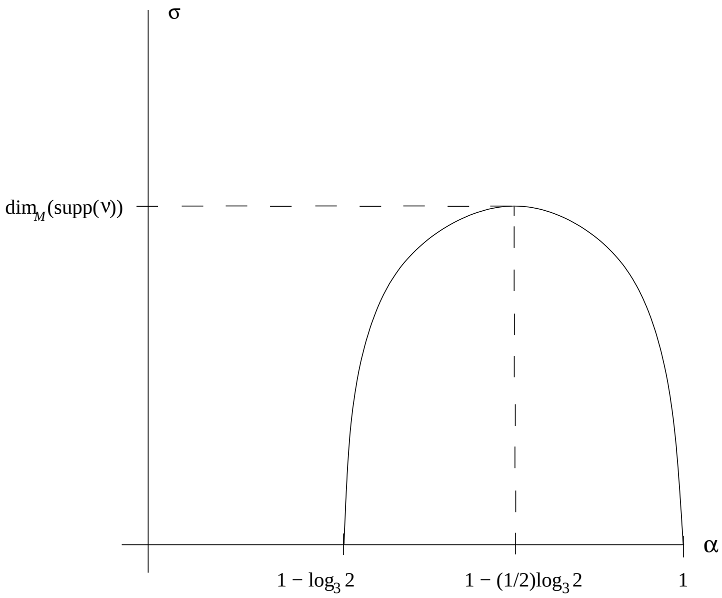
where k_1 and k_2 are relatively prime non-negative integers such that $k_1 < k_2$.



Construction of the partition zeta function $\zeta_{\mathfrak{P}}^{\nu}(\alpha(1,2),s)$. The solid black blocks correspond to the closed intervals with regularity $\alpha(1,2)$. In general,

$$\zeta_{\mathfrak{P}}^{\nu}(\alpha(k_1,k_2),s) = \sum_{n=1}^{\infty} \binom{nk_2}{nk_1} 3^{-nk_2 s},$$

for $\mathrm{Re}s$ large enough.

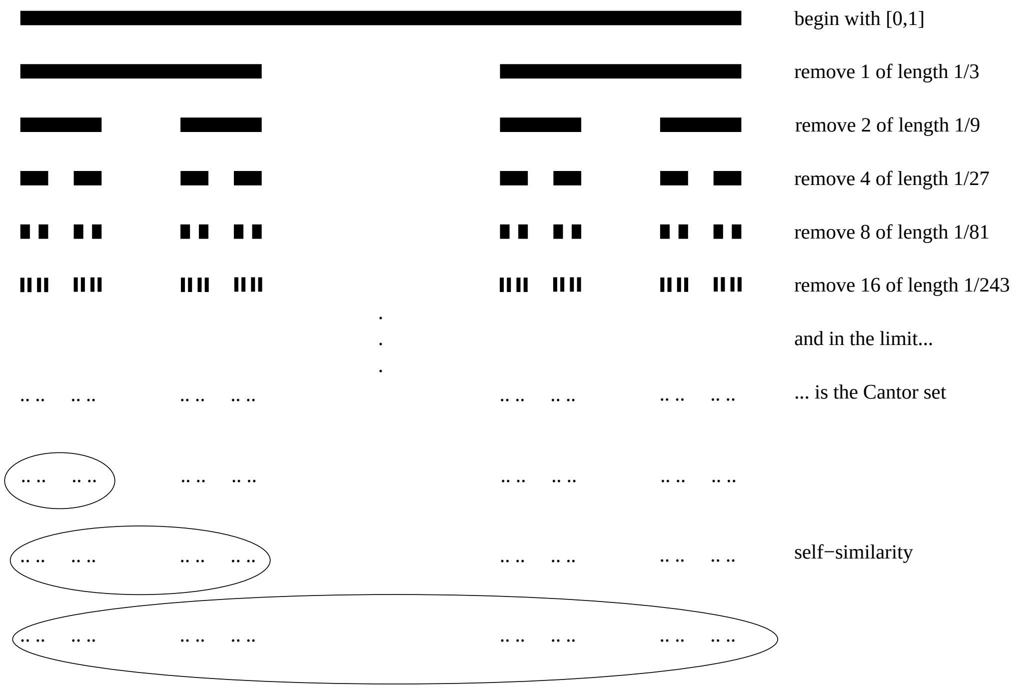


The abscissa of convergence function σ for the measure ν has the form

$$\begin{aligned}\sigma(\alpha) &= \frac{(\alpha-1)}{\log_3 2}\cdot\log_3\left(\frac{-(\alpha-1)}{\log_3 2}\right)\\ &\quad-\left(1+\frac{(\alpha-1)}{\log_3 2}\right)\cdot\log_3\left(1+\frac{(\alpha-1)}{\log_3 2}\right).\end{aligned}$$

The maximum of σ is attained at $\alpha=\alpha(1,2)=1-(1/2)\log_3 2$ and this value coincides with the box dimension of the support of the measure ν. That is,

$$\begin{aligned}\dim_B(supp(\nu)) &= \max\{\ \sigma(\alpha)\ \mid\ \alpha=\alpha(k_1,k_2)\}\\ &= \log_3 2.\end{aligned}$$



The Cantor set is constructed by removing 2^{n-1} open intervals of length $1/3^n$ for every $n \in \mathbb{N}$ from the unit interval $[0, 1]$.

References

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- Website for Mandelbrot set applet: http://math.hws.edu/xJava/MB/