A Tapestry of Complex Dimensions

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May 7th, 2009
Multifractal Measures and Their Spectra

Multifractals are used to model natural phenomena which have very irregular structure. The distribution of stars in a galaxy, the distribution of minerals in a mine, and the formation of lightning are considered to be multifractal and are mathematically modeled by measures.

**GOAL:** In this talk, we’ll find most members of the family of multifractal zeta functions for a simple measure $\sigma$ and compute the corresponding multifractal spectrum and complex dimensions.
2 Measures and Regularity

The measure we consider acts on closed subintervals $U$ of the unit interval $[0, 1]$. Multifractal zeta functions are parameterized regularity, which connects the size of an interval with its mass.

**Definition 2.1.** The regularity $A(U)$ of an interval $U$ with respect to the measure $\beta$ is

$$A(U) = \frac{\log \beta(U)}{\log |U|},$$

where $|U|$ is the length of $U$.

Regularity $A(U)$ is also known as the coarse Hölder exponent $\alpha$ which satisfies

$$|U|^\alpha = \beta(U).$$
3 Multifractal Zeta Functions

Collecting the lengths of the intervals $K_p^n(\alpha)$ according to their regularity $\alpha$ allows us to define the multifractal zeta functions.

**Definition 3.1.** The **multifractal zeta function** of a measure $\mu$, sequence os scales $N$ and with associated regularity value $\alpha$ is

$$\zeta_{\mu}^N(\alpha, s) = \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} |K_p^n(\alpha)|^s,$$

for $\text{Re}(s)$ large enough.

What are these $K_p^n(\alpha)$, exactly? Ask me later.
4 Simple $\sigma$

Let $\mathcal{N} = \{3^{-n}\}_{n=1}^{\infty}$ and

$$\sigma = \sum_{j=1}^{\infty} 3^{-j} \delta_{3^{-j}}.$$

**goal:** Find all the regularity values attained by $\sigma$ with intervals $U$ that have length in $\mathcal{N}$.

Figure 2: Approximation of the measure $\sigma$. 
The positive values of $\sigma(U)$ are obtained in one of the two following ways:

- **Case 1:** $U$ contains exactly one point-mass of size $3^{-j}$ where $j \leq N$.

- **Case 2:** $U$ contains two or more point-masses, necessarily including the point-mass $3^{-N}$. If any other smaller point-mass $3^{-p}$ is also contained in $U$, so are all point-masses $3^{-j}$ between $3^{-p}$ and $3^{-N}$ (i.e., $N \leq j \leq p$). That is, $U$ contains any finite or infinite sequence of point-masses $\{3^{-j}\}_{j=N}^{p}$, where $p > N$.

**Lemma 4.1.** For the measure $\sigma$ and sequence of scales $\mathcal{N} = \{3^{-n}\}_{n=1}^{\infty}$, the possible finite regularity values of $U$ where $|U| = 3^{-N}$ for some fixed $N \in \mathbb{N}$ are:

$$\alpha(m_1, m_2) = \frac{\log 3^{-m_1 n}}{\log 3^{-m_2 n}} = \frac{m_1}{m_2},$$

$$\alpha_N(p) = \log \left( \frac{(3^p-N+1) - 1}{2} \right) + \frac{p + 1}{N},$$

where $m_1 < m_2$ and $(m_1, m_2) = 1$ for $m_1, m_2 \in \mathbb{N}$, and $p \in \mathbb{N} \cup \{\infty\}$ and $p > N$. These regularity values are all distinct from one another.
Figure 3: Approximation of $\sigma$ and the construction of $\zeta_N(1/2, s)$.

The solid black bars represent the $K_p^n(\alpha)$ with $\alpha = 1/2$ that generate the terms of the multifractal zeta function. For $\alpha(m_1, m_2) = m_1/m_2$, only stages at multiples of $m_2$ have intervals with the correct regularity, hence the other stages are skipped.
The breakdown for all possible regularity values associated with the measure $\sigma$ and sequence $\mathcal{N}$ provided by Lemma 4.1 allows for the complete breakdown of all the possible multifractal zeta functions of $\sigma$ with $\mathcal{N}$.

**Theorem 4.2.** For the measure $\sigma$ and sequence $\mathcal{N} = \{3^{-n}\}^\infty_{n=1}$, the nontrivial multifractal zeta functions have the following forms*:

$$
\zeta^\sigma_{\mathcal{N}} \left( \frac{m_1}{m_2}, s \right) = \sum_{j=1}^{\infty} \left( \frac{2}{3^{m_2j}} \right)^s = \left( \frac{2}{3^{m_2}} \right)^s \left( \frac{1}{1 - 3^{-m_2 s}} \right),
$$

$$
\zeta^\sigma_{\mathcal{N}} (1, s) = \left( \frac{5}{9} \right)^s + \sum_{j=1}^{\infty} \left( \frac{2}{3^{j+2}} \right)^s
$$

$$
= \left( \frac{5}{9} \right)^s + \left( \frac{2}{27} \right)^s \left( \frac{1}{1 - 3^{-s}} \right),
$$

where $m_1 < m_2$ and $(m_1, m_2) = 1$ for all $n, m_1,$ and $m_2 \in \mathbb{N}$. For all other regularity values, the corresponding multifractal zeta functions are entire.

*This list is incomplete, technically.
5 Complex Dimensions

Remark 5.1. Similar to what is done with geometric zeta functions, under appropriate conditions it is assumed that, as a function of $s \in \mathbb{C}$, $\zeta^\mu_N(\alpha, s)$ admits a meromorphic continuation to an open neighborhood of a window $W$. We may then consider the poles of these zeta functions as complex dimensions.

Definition 5.2. For a measure $\mu$, sequence $N$ which tends to zero and regularity value $\alpha$, the set of complex dimensions with parameter $\alpha$ is given by

$$D^\mu_N(\alpha, W) = \{\omega \in W \mid \zeta^\mu_N(\alpha, s) \text{ has a pole at } \omega\}.$$
The formulas for the multifractal zeta functions provided by Theorem 4.2 immediately yield the following collections of complex dimensions with all regularity values $\alpha$.

**Corollary 5.3.** Under the assumptions of Theorem 4.2, the complex dimensions with parameter $\alpha$ of the measure $\sigma$ and sequence $\mathcal{N} = \{3^{-n}\}^\infty_{n=1}$ are the poles of the multifractal zeta function $\zeta_{\mathcal{N}}^\sigma(\alpha, s)$. For the nontrivial values of $\alpha$ described in Theorem 4.2,

$$\mathcal{D}_{\mathcal{N}}^\sigma(m_1/m_2, W) = \left\{ \omega \in W \mid \omega \text{ is a pole of } \zeta_{\mathcal{N}}^\sigma \left( \frac{m_1}{m_2}, s \right) \right\}$$

$$= \left\{ \frac{2\pi iz}{m_2 \log 3} \right\}_{z \in \mathbb{Z}},$$

$$\mathcal{D}_{\mathcal{N}}^\sigma(1, W) = \left\{ \omega \in W \mid \omega \text{ is a pole of } \zeta_{\mathcal{N}}^\sigma (1, s) \right\}$$

$$= \left\{ \frac{2\pi iz}{\log 3} \right\}_{z \in \mathbb{Z}},$$

for appropriate windows $W$. 
Remark 5.4. All of the poles above have real part zero. Consider the space $\mathbb{R} \times \mathbb{C}$, where to $\mathbb{R}$ we associate the collection of finite regularity values $\alpha$ and to $\mathbb{C}$ we associate the corresponding complex dimensions with parameter $\alpha$. For the measure $\sigma$ and sequence $\mathcal{N}$, the full family of complex dimensions of all $\alpha$ is a dense subset of the strip in $\mathbb{R} \times \mathbb{C}$ given by $[0,1] \times \{ s \in \mathbb{C} \mid \text{Re}(s) = 0 \}$. Specifically, we get the set

$$\{ (\alpha, \omega) \mid \alpha \in [0,1] \cap \mathbb{Q}, \text{Re}(\omega) = 0, \text{Im}(\omega) = \frac{2\pi iz}{k \log 3} \text{ for } k \in \mathbb{N} \}.$$ 

This is the tapestry of complex dimensions corresponding to the measure $\sigma$. 
References


[14] M. L. Lapidus and J. A. Rock, Towards zeta functions and complex dimensions of mul-
tifractals, *Complex Variables and Elliptic Equations*, special issue dedicated to fractals, in press. (See also: Preprint, Institut des Hautes Etudes Scientifiques, IHES/M/08/34, 2008.)


