Section 9

27a) Prove that every permutation in $S_n$ can be written as a product of at most $n-1$ transpositions.

Proof: Assume $n \geq 3$

Let $A = \{1, 2, 3, \ldots, n-1, n\}$

Let $\tau \in S_n$ \(\tau = (1 \ 2 \ 3 \ \cdots \ n-1 \ n) = (1 \ n)(1 \ n-1) \cdots (1 \ 3)(1 \ 2)\) \(\text{by the def. of transpositions we can repeat one element.}\)

Let $T \in S_n \implies T$ has $n$ disjoint cycles with $n \geq 2$

Let $l_\ell$ be the length of each disjoint cycle, \(\ell \in 1, 2, 3, \ldots, m\)

Notice \[\sum_{\ell=1}^{m} l_\ell \leq n \quad --- (1)\]

Now

In the first cycle we can have at most $l_1 - 1$ transpositions

In the second cycle we can have at most $l_2 - 1$ transpositions

In the $m$th cycle we can have at most $l_m - 1$ transpositions

Thus the total number of transpositions is

\[(l_1 - 1) + (l_2 - 1) + \cdots + (l_m - 1)\]

\[= l_1 + l_2 + \cdots + l_m - (1 + 1 + \cdots + 1)\]

\[= \sum_{i=1}^{m} l_i - m \leq n - m \quad \text{by (1)}\]

Since $m \geq 2$ then $n - 1 > n - m$

\[\therefore\text{ Every permutation in } S_n \text{ with } n \geq 3 \text{ can be written as a product of at most } n - 1 \text{ transpositions}\]