Back in Section 2.1, we computed the slope of the tangent line to the curve \( y = x^3 \) at the point (1, 1). Starting with a second point on the curve close to (1,1) and writing this point as \((x, x^3)\), we computed the slope of the resulting secant line as follows:

\[
\text{Slope of secant line} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{x^3 - 1}{x - 1}
\]

The slope of the tangent line was found by pushing \( x \to 1 \) in the second point. Then

\[
\text{Slope of tangent line at } (1, 1) = \frac{d}{dx} y = \frac{d}{dx} x^3 = 3x^2
\]

But recall that \( f(x) = x^3 \) and \( f(1) = 1 \). Thus,

\[
\text{Slope of tangent line at } (1, 1) = \frac{3}{1} = 3
\]

If instead at \( x = 1 \) we use \( x = a \), then we obtain a general formula for the slope of the tangent line at \( x = a \).

**Definition 1.** The *slope of the tangent line* to \( y = f(x) \) at \( x = a \) is the limit of the slope of the secant lines connecting \((a, f(a))\) and \((x, f(x))\) as \( x \to a \):

\[
\text{Slope} = \lim_{x \to a} \frac{y - f(a)}{x - a} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)
\]

if this limit exists.

The *tangent line* to \( y = f(x) \) at \( x = a \) is the line through \((a, f(a))\) whose slope is found in the formula above.

**Problem 1.** Compute the slope of the tangent line to \( f(x) = x^2 \) at the point \((7, 49)\). Then find an equation for the corresponding tangent line.
There is another way to write the formula for the slope of a tangent line, which we will frequently use. Set \( h = x - a \). Then

\[
x =
\]

\[
f(x) =
\]

\( x \to a \) can be rewritten as

The slope of the tangent line to \( y = f(x) \) at \( x = a \) can then be written as

**Problem 2.** Use this new formula to compute the slope of the tangent line to \( f(x) = x^2 \) at \( x = 7 \).

In Section 2.1, we also computed instantaneous velocities of an object by using a procedure similar to that for finding tangent line slopes. We computed average velocities over smaller and smaller time intervals by moving the ending time closer and closer to the starting time. So we define instantaneous velocity at time \( t = a \) in the same way as slopes of tangent lines.

**Definition 2.** Let \( f(t) \) be the position of an object at time \( t \). Then the **instantaneous velocity** of the object at time \( t = a \), denoted \( v(a) \), is given by
In fact, we can look at any rate of change in this fashion. Suppose \( f(x) \) is a function, and let’s start at \( x = x_1 \). If the value of \( x \) moves changes from \( x_1 \) to \( x_2 \), then the value of \( f \) changes from \( f(x_1) \) to \( f(x_2) \). We can then compute the **average rate of change** of \( f \) from \( x_1 \) to \( x_2 \) with the formula

\[
\text{Average rate of change} = \frac{\text{Change in } f(x)}{\text{Change in } x} = 
\]

If we want to find the instantaneous rate of change of \( f \) at \( x = x_1 \), then we let \( x_2 \rightarrow x_1 \). Then . . .

**Definition 3.** The **instantaneous rate of change** of \( f \) at \( x = x_1 \) is

\[
\text{Instantaneous rate of change} = 
\]

Since instantaneous rates of change are defined as limits of change in \( f(x) \) divided by change in \( x \), we can determine the units of the instantaneous rate of change:

\[
\text{Units of instantaneous rate of change} = 
\]

**Problem 3.** Suppose the temperature of an object at time \( t \) minutes is given by \( f(t) = \frac{10}{t+1} \) degrees Celsius. Find the instantaneous rate of change of the temperature at \( t = 2 \) minutes.

The primary limit we have constructed in this section is extremely important. It is so important that we give it its own name...
Definition 4. The derivative of $f$ at $x = a$, written as $f'(a)$ and read as “$f$ prime of $a$,” is

$$f'(a) =$$

Equivalently, we also say that

$$f'(a) =$$

Let’s compute a few derivatives to get a feel for what’s going on.

**Problem 4.** Compute the derivative of $f(x) = 4 + 9x - 2x^2$ at $a = 3$.

**Problem 5.** Compute the derivative of $f(x) = \sqrt{x - 7}$ at $a = 16$. 
Based on what we have discussed in this section, there are several interpretations for what the derivative represents. Keep these equivalent statements in mind. We will use them throughout the rest of the term.

The derivative \( f'(a) \) represents...

- The slope of the tangent line to \( y = f(x) \) at \( x = a \).
- The instantaneous velocity of the object at time \( a \), if \( f \) is the object’s position.
- The instantaneous rate of change of \( f \) at \( x = a \).

Looking at the temperature example we did in Section 3.1, if the temperature of an object at time \( t \) (in minutes) is given by \( f(t) = \frac{10}{t+1} \) degrees Celsius, then \( f'(2) = -\frac{10}{9} \degree C/min \).

We can also determine the units of \( f'(a) \). Since derivatives are instantaneous rates of change, they both have the same units.

Units of \( f'(a) = \)

Derivatives do have many practical contexts. One that we have already seen involves instantaneous velocity: if \( s(t) \) represent the position of an object, then \( s'(t) \) is its instantaneous velocity. But if a function \( f(x) \) represents some other physical quantity, then \( f'(a) \) represents how fast that quantity is changing at \( x = a \).

For example, suppose that the total cost \( C \) of repaying a loan at an annual interest rate of \( r\% \) is given by some function \( C = f(r) \). Then \( f'(a) \) represents how quickly the cost changes as the interest rate changes. For example, \( f'(10) = 1200 \) means that as the interest rate is raised through 10%, then the cost increases at a rate of $1200 per annual percentage point.
Problem 6. Suppose the cost (in dollars) of producing $x$ cars is given by $C(x) = 5000 + 10x + 0.05x^2$. What does $C'(20)$ represent? Describe it in one or two sentences.

Problem 7. Sketch the graph of a function $f$ for which $f(0) = 0$, $f'(0) = 1$, $f'(2) = 0$, and $f'(4) = -3$.

Problem 8. Often, it will be useful to compute the derivative of $f$ at an arbitrary value $x = a$. In such an instance, the derivative will contain $a$ in the final answer. Compute the derivative of $f(x) = x^3$ at $x = a$. 
Problem 9. Compute the derivative of $f(x) = \sqrt{4-x}$ at $x = a$.

With a general formula for $f'(a)$, we can compute the equation of a tangent line to $f(x)$ at any appropriate value of $x = a$. For example, let $a = -5$. Then we can make the following computations, using the function $f(x) = \sqrt{4-x}$ from the last example.

- Point: $(-5, f(-5)) = $  
- Slope: $f'(-5) = $

Now that you have the slope and a point, what is the equation of the tangent line?