Harmonic Motion Equations with Related Applications

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Received 16 April, 2019

Abstract

The paper will focus on the understanding of harmonic functions and their relationship with its applications. Particularly, we will be investigating its foundations in differential equations and numerical methods and its effect on the expression of mathematical models. Using ordinary differential equations along with numerical methods further analyzes various mathematical applications often seen in spring problems and in the oscillations of objects. Thus, we begin on a comprehensive understanding of harmonic functions and their relationship with its applications, particularly investigating its foundations in four different types of ordinary differential equations. The equations include the homogenous, non-homogenous, resonance and damped equations. In order to create a thorough understanding, we must take a closer look at systems which contain an oscillatory motion. In addition, important concepts such as amplitude and frequency will be incorporated to fluently comprehend the background information given and further explain the applications of harmonic motion equations. Through the implementation of MATLAB, visual representations of the equations and their coefficients were created and analyzed to further determine the importance in the variance of coefficients. Furthermore, being able to thoroughly comprehend how each equation varies from one another builds a correlation on the emphasis of real-world applications on simplistic and ordinary mathematical concepts and equations. In conclusion, the resulting information may be used as an invaluable resource for students and construct an oscillatory system varying from elementary mathematics to higher level applicational mathematical equations.

Keywords: Ordinary Differential Equations, harmonic motion, frequency, oscillations, resonance, damping force, homogenous, non-homogenous.

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Introduction

The purpose of this research is to demonstrate the importance of mathematics and mathematical modeling. Through the use of Differential Equations and Harmonic Motion Equations, we will further investigate the impact of external forces and their role oscillating objects. By investigating the motion of a system, it will shift focus on the sinusoidal curve when external forces are acting upon it.

Now, we will focus on some important terms that are necessary to grasp the idea of harmonic motion. A simple harmonic motion is defined to be an oscillation motion under a retarding force proportional to the amount of displacement from an equilibrium position [9]. A few ideas that we will explore include resonance, frequency and oscillation. Resonance is defined as the condition in which an object is subjected to an oscillating force having a frequency close to its own natural frequency. Frequency is described as the number of waves that pass a fixed place in a given amount of time. Finally, oscillation is a repetitive back and forth motion at a constant velocity [9].

In this paper, we will investigate four different differential equations to analyze the movement of a harmonic system. These equations are:

(1) $x''(t) + k^2x(t) = 0$

(2) $x''(t) + k^2x(t) = cos(wt)$, *where* $w \neq k$

(3) $x''(t) + k^2x(t) = cos(wt)$, *where* $w = k$

(4) $x''(t) + 2\lambda x'(t) + k^2x(t) = cos(wt)$

The first three equations are described as simple harmonic motion. They analyze the behavior of an oscillating object from a theoretical point of view. The fourth equation is also a mass system, but includes a damping factor which affects the behavior of the equation. By adding the damping force, it creates a more accurate and realistic movement of the system.

One of the most important aspects of these equations are the variables. In order to have a firm grasp on the meaning of the equations, we must break down every variable and look at its meaning. This will allow us to understand every part of the equation. In the appendix, there will be a chart in which every term and coefficient will be explained in detail.

Origin of the Homogeneous Equation

It is necessary to understand the basis of these equations and their origin in order to understand the impact of each equation. The foundational differential equation is reliant upon two simple free-motion equations: Hooke's Equation and Newton's Second Law. They are important because the differential equation, $x''(t) + k^2x(t) = 0$ is a result of a combination between these two foundational equations. We will proceed by exploring both Hooke's Equation and Newton's Second Law and how they are use to create the introductory differential equation of motion.

Hooke's Law states that the amount stretched is proportional to the restoring force [5]. When coming up with these equations, there are certain criteria that have to be met to show that it is a simple harmonic motion (SHM) equation. That is:

1) There is a restoring force proportional to displacement from equilibrium
2) Potential energy is proportional to the square of the displacement
3) Period/Frequency is independent of the amplitude
4) Position, velocity, and acceleration are sinusoidal in time (modeled by sine and cosine)

First, we will take a look at Hooke's Law and how it pertains to SHM equations. With Hooke's Law, we need a distance of displacement, x(t), and a spring constant that exerts an equal and opposite force. The equation for Hooke's Law is:

$$F = -s \cdot x(t)$$

where s is our spring constant acting in the opposite direction on the force being applied. The other variable, x(t), is our restoring force proportional to the displacement of equilibrium.

Now we will take a look at Newton's Second Law. Newton's Second Law pertains to the behavior of objects for which all existing forces are not balanced. That is, the acceleration of an object is dependent upon two variables — the net force acting upon the object and the mass of the object. [9] This is where we get the formula:

$$F = ma$$

It is also important to note that acceleration is the second derivative of displacement with respect to time. That is

$$a = \frac{dv}{dt} \text{ and } v = \frac{ds}{dt}$$

Notice how in both laws, the equations for both are dealing with force, F. Since both of the equations are pertaining to force, we are able to set them equal to each other and get:

$$F = m \cdot x''(t)$$

$$m \cdot x''(t) = -s \cdot x(t)$$

Dividing both sides by the mass we get:

$$x''(t) = \frac{-s}{m} \cdot x(t)$$

Since our spring constant is equal to Newtons over meters we have the

$$s = \frac{N}{m}$$

Notice that the units for Newtons is (1 kg)· $(\frac{m}{s^2})$. Then we have the equation

$$s = \frac{kg}{s^2}$$

$$s = \frac{\frac{kg}{s^2}}{kg}$$

Which simplifies down to

$$s = \frac{1}{s^2}$$

Now we will introduce a new variable, k measured in units of (1/s). Thus, we come up with the new term k^2= ($1/s^2$) and introduce this constant into the original equation, getting

$$x''(t) = -k^2 x(t)$$

Upon adding our new term to the other side, we get our final product that is a simple harmonic motion equation that looks like:

$$x''(t) + k^2 x(t) = 0$$

In combining Hooke's Law and Newton's Second Law of Motion, we create an equation that is identical to the homogeneous differential equation. This equation is also known as the Harmonic Oscillator equation, which describes the theoretical, or natural movement of the object without any outside forces affecting its movement. As described above, $x(t)$ describes the displacement of the object over time, while $x'(t)$ describes the velocity of the object over a given time period.

Homogeneous Equation

Our Homogeneous Equation is $x''(t) + k^2 x(t) = 0$. To find the solution of the homogeneous equation, we must first find the auxiliary equation related to the harmonic motion equation. In the auxiliary equation, we assign every nth-order derivative with a nth-order exponent. Thus, $x''(t)$, second derivative, will be written as n^2 and x(t) will be written as 1. Thus, the auxiliary equation will be:

$$n^2 + k^2 = 0$$

Our solutions to the auxiliary equation will be complex, equaling

$$n_1 = ki \text{ and } n_2 = -ki.$$

Now that we have the solutions, we can create a general solution with our complex numbers. Our general solution for our differential equation will be:

$$x(t) = c_1 cos(kt) + c_2 sin(kt)$$

Through our general solution, we are able to see that the natural frequency of the object is k. This means that there are no external forces acting on the system. In reality, this equation treats the object as if it were inside a vacuum. The homogeneous solution becomes a building block for the non-homogeneous solution. In looking at the solution to the homogeneous equation, we can see that there are many variables that need to be taken into consideration. These include the arbitrary constants c_1 and c_2. When these two variables have different values, they can change the expression of the equation. There are four different possible cases include:

1) $c_1 = c_2$
2) $c_1 > c_2$
3) $c_1 < c_2$
4) $c_1 = -c_2$

As mentioned earlier, we observe each case to account for each possibility in the real world. As initial conditions change from scenario to scenario we will verify that the solutions will behave similarly for the homogenous case. Later we will discuss the non-homogeneous case in which the amplitude will vary slightly. As one can see in the graph below, the amplitude in the homogeneous equation stays constant, with no variations to the size in amplitude in each wave.

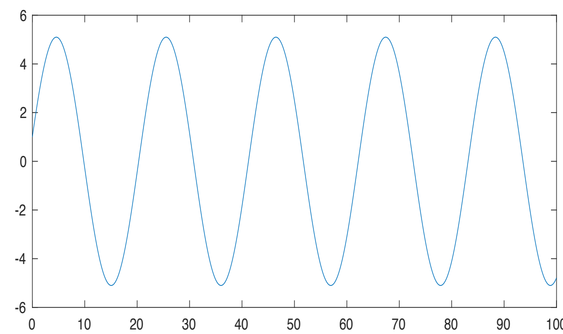

Figure 1. Graph of Homogeneous Equation

The Non-Homogeneous Equation

The non-homogenous equation is $x''(t) + k^2 x(t) = cos(wt)$, where $w \neq k$. As one can see, it is very similar to the homogeneous equation, except there is an outside force acting upon the system ($\cos(wt)$). This equation will analyze the motion of the object when it is close to resonance. This focuses on external forces that may affect its frequency. The particular solution to the equation will be an equation containing sines and cosines. Since the equation is very similar to the homogeneous equation, our solution will be almost identical to that of the homogeneous solution, but with the addition of the particular solution. Thus, our solution will look like:

$$x(t) = c_1 cos(kt) + c_2 sin(kt) + x_p$$

The particular solution can be found using the method of undetermined coefficients.

Let

$$x_p = Asin(wt) + Bcos(wt),$$

where x_p is the particular solution to the non-homogenous equation. The particular solution is necessary because we no longer have a homogeneous equation. Since our equation is equal to cos(wt), we must consider its derivative as well. That is why

$$x_p = Asin(wt) + Bcos(wt).$$

Then, we take the first and second derivatives of the particular solution

$$x_p{}' = Awcos(wt) - Bwsin(wt)$$

$$x_p{}'' = -Aw^2 sin(wt) - Bw^2 cos(wt)$$

Next, we plug the particular solution into the differential equation, so we will have,

$$x_p''(t) + k^2 x_p(t) = -Aw^2 sin(wt) - Bw^2 cos(wt) + k^2[Asin(wt) + Bcos(wt)] = cos(wt)$$

$$-Aw^2 sin(wt) - Bw^2 cos(wt) + Ak^2 sin(wt) + Bk^2 cos(wt) = cos(wt)$$

$$Asin(wt)[k^2 - w^2] + Bcos(wt)[k^2 - w^2] = cos(wt)$$

Now, set the cosines on each sides of the equation equal to one another and the sines equal to each other. Thus,

$$Asin(wt)[k^2 - w^2] = 0 \text{ and } Bcos(wt)[k^2 - w^2] = cos(wt)$$

Therefore,

$$A = 0 \quad \text{and} \quad B = \frac{1}{k^2 - w^2}$$

So,

$$x_p = \frac{1}{k^2 - w^2} \cos(wt)$$

Thus, the solution for the Non-Homogenous case is:

$$x(t) = c_1 cos(kt) + c_2 sin(kt) + \frac{1}{k^2 - w^2} \cos(wt)$$

The non-homogeneous equation is very important because it is considered a more realistic situation. By adding the $cos(wt)$on the opposite side of the equation, it begins to integrate the outside forces that affect the system. By incorporating the cos(wt) to demonstrate the outside forces, it converts the differential equation from a theoretical viewpoint to an application problem. Below, is a graph of the non-homogenous case so that one may have a visual representation of what the equation signifies.

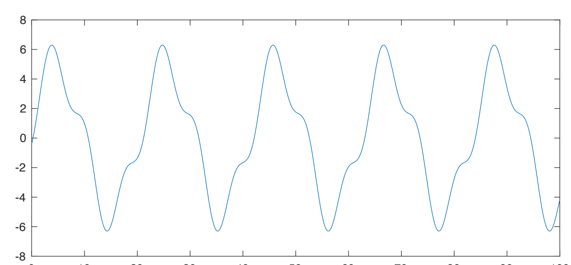

Figure 2. Graph of Non-Homogenous Case

Resonance Equation

Our third equation focuses when the system reaches resonance. If the system's external forces cause it to meet the natural resonance of the system, it can cause the system to move uncontrollably. Since there is no particular method that can solve when $w = k$, we will approximate its solution through limits. We will begin by taking the limit of the solution of the non-homogenous equation and using it as our base for this solution. Thus,

$$\lim_{w \to k} \left(c_1 cos(kt) + c_2 sin(kt) + \frac{1}{k^2 - w^2} \cos(wt) \right).$$

Our main priority will be evaluating

$$\lim_{w \to k} \left(\frac{1}{k^2 - w^2} \cos(wt) \right).$$

Directly evaluating our limit would result in an indeterminant form. Thus, L'Hospita'ls rule is necessary to solve the limit. So,

$$\overset{\text{LH}}{=} \lim_{\omega \to k} \frac{-tsin(\omega t)}{-2\omega} = \frac{tsin(kt)}{2k}$$

Therefore, our general solution for the Resonance Equation is:

$$x(t) = c_1 cos(kt) + c_2 sin(kt) + \frac{tsin(kt)}{2k}$$

We consider the case where w = k because at this point, the oscillations of the object have achieved resonance. Once the oscillations begin to increase, they continuously become large. In a spring system this means they have gone beyond their elastic limit. Thus, in order to stop the oscillations from achieving resonance, we need a damping force.

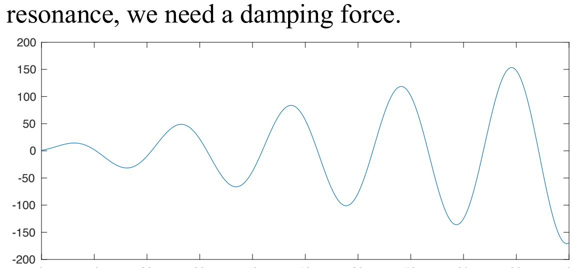

Figure 3. Graph of Resonance Equation

Introduction to the Damped Equation

The fourth equation also inquests the movement of the system, but includes a damping force, unlike the other equations. A damping force is a retarding force such that over time the oscillations of the moving object become increasingly smaller by reducing the amplitude. The damping force is important in modeling realistic systems, as it forces the system to return to equilibrium. Since the system is not suspended in a vacuum, there will be at least one damping force due to the surrounding medium. The damping force creates three different scenarios for the equation: overdamped ($\lambda^2 - k^2$>0), critically damped ($\lambda^2 - k^2$= 0), and underdamped ($\lambda^2 - k^2$< 0).

While considering the external forces of the equation, we must also consider the effect of the damping force on the system. In most cases, the damping force exists in order to force the system back into equilibrium. Without the damping force, the system would continue to oscillate indefinitely. The damping force may be expressed through friction, shock absorbers, and many other things that help return the motion to equilibrium.

In order to determine if the fourth equation is overdamped, critically damped, or underdamped, we must look at the auxiliary equation,

$$x''(t) + 2\lambda x'(t) + k^2 x(t) = 0$$

And find its solution. The damping force is denoted by 2. This force is what enables the mass system to approach equilibrium over time, thus allowing the oscillatory displacement of the object to become minimal. Once we set up and find the solutions to our auxiliary equations, we get:

$$n^2 + 2\lambda n + k^2 = 0$$

$$n = \frac{-2\lambda \pm \sqrt{4\lambda^2 - 4k^2}}{2} = -\lambda \pm \sqrt{\lambda^2 - k^2}$$

Now that we have the solutions to the characteristic equation for (4), we can now figure out if the forces acting upon our equation are over, critically, or underdamped. To do this, we focus on the discriminant, $\lambda^2 - k^2$.

Overdamped Equation

If $\lambda^2 - k^2$> 0, we refer to this situation as overdamped. One real life application to overdamped motion is through the modelling automatic paraplegic doors. The design of these types of doors includes a spring, which is damped. This damping allows the door to open and close at a chosen rate [8]. This damping is overdamped because the opening and closing is not an oscillatory motion. When the motion is overdamped, we notice the damping constant, $\beta = 2\lambda m$, is larger when compared to the spring constant, k, thus resulting in a smooth, generally non-oscillating curve. The solutions are modelled by the following,

$n_1 = -\lambda + i\sqrt{k^2 - \lambda^2}$ and $n_2 = -\lambda - i\sqrt{k^2 - \lambda^2}$

$$x(t) = c_1 e^{n_1 t} + c_2 e^{n_2 t}$$

$$x(t) = c_1 e^{(-\lambda + i\sqrt{k^2-\lambda^2})t} + c_2 e^{(-\lambda - i\sqrt{k^2-\lambda^2})t}$$

Thus, the case of overdamped occurs results in the above characteristic equation x(t)and can be used as a basis in modeling applications such as the design of paraplegic doors and more.

Critically Damped Equation

If $\lambda^2 - k^2$= 0, then we say that it is critically damped. When an object is critically damped, the curve will graphically cross the horizontal axis once, achieve maximum amplitude, and over time decrease to 0. Properties of the curve that we notice are that it is continuous and contains oscillations with infinitesimally small amplitudes. Physical systems that exhibit critically damped motion are shock absorbers on cars. The springs in the shock absorber will react to any shock, such as driving over a speed or pot hole, due to the spring coming back to equilibrium at a fast rate [4]. A noticeable property of the damping force associated with being critically damped slight is any decrease in the force can result in an oscillatory motion. The solutions for the critically damped scenario are modeled by the following,

$$n_1 = -\lambda \text{ and } n_2 = -\lambda$$

$$x(t) = c_1 e^{-\lambda t} + c_2 e^{-\lambda t}$$

Thus, our characteristic equation is shown above and can be used in the modelling of the functionality of shock absorbers and many other applications.

Underdamped Equation

Finally, if $\lambda^2 - k^2$< 0, then the equation is underdamped. Physically, what is happening to the object moving in underdamped motion is the amplitude of the object is decreasing over time. Physical systems that experience underdamped motion are suspension bridges, which exhibit this type of damping due to their design. The suspension bridge is designed with suspension cables made of springs, and are designed with a piece called a girder which helps weigh down the bridge. Because of external forces acting on the it, the bridge will swing in an oscillatory motion, thus the girder acts as the damping force which slowly decreases these oscillations to the point of becoming close to zero [7].

When underdamped, we notice the damping constant, β, is small when compared to the spring constant, k. The most important phenomena associated with underdamped motion is that as $t \longrightarrow \infty$ the amplitude of motion, $e^{-\lambda t} \to 0$. The solutions for the underdamped scenario are modelled with the following,

$n_1 = -\lambda + i\sqrt{k^2 - \lambda^2}$ and $n_2 = -\lambda - i\sqrt{k^2 - \lambda^2}$

$$x(t) = e^{-\lambda t}[c_1 \cos\left(\sqrt{k^2 - \lambda^2}\ \mathrm{t}\right) + c_2 \sin\left(\sqrt{k^2 - \lambda^2}\ \mathrm{t}\right)]$$

This final case of damping is modelled above and generally produces the most oscillatory motion out of the three cases of damping, this is due to the sine and cosine terms within the characteristic equation. This can be used to model many different mass systems such as suspension bridges or pendulums.

Non-Homogenous Cases

We have examined the homogeneous model (4) and found solutions for each type of damping: overdamped, critically damped, and underdamped. However, much like the homogeneous equation (1), this case is mostly theoretical and does not account for any external forces acting upon the object. Thus, we consider the more applicable, non-homogeneous equation (4)

$$x''(t) + 2\lambda x'(t) + k^2 x(t) = cos(wt)$$

Since we have solved for the characteristic equation above when solving the homogeneous solution. We will now find the particular solution by letting

$$x_p(t) = Acos(wt) + Bsin(wt)$$

By substituting in the first and second derivatives of $x_p(t)$ we obtain the following equation

$$-w^2 Acos(wt) - w^2 Bsin(wt) - 2\lambda wAsin(wt) + 2\lambda wBcos(wt) + k^2 Acos(wt) + k^2 Bsin(wt) = \cos(wt)$$

Next, we must solve for the coefficients A and B through the method of undetermined coefficients we have the following system of equations.

$$(k^2 - w^2)A + 2\lambda wB = 1$$

$$-2wA + (k^2 - w^2)B = 0$$

The solution to the system is $A = \frac{k^2-w^2}{(k^2-w^2)^2+4w^2\lambda^2}$ and $B = \frac{2\lambda w}{(k^2-w^2)^2+4w^2\lambda^2}$. We must note though, $k \neq w \neq 0, and\ \lambda \neq 0$. Thus, the solutions to the non-homogeneous equation follow the following cases:

Case I: $\lambda^2 - k^2 > 0$ (overdamped)

$$n_1 = -\lambda + \sqrt{\lambda^2 - k^2} \qquad n_2 = -\lambda - \sqrt{\lambda^2 - k^2}$$

$$x(t) = c_1 e^{n_1 t} + c_2 e^{n_2 t}$$

$$x(t) = c_1 e^{(-\lambda+\sqrt{\lambda^2-k^2})t} + c_2 e^{(-\lambda-\sqrt{\lambda^2-k^2})t}$$

Case II: $\lambda^2 - k^2 = 0$ (critically damped)

$$n_1 = -\lambda \qquad n_2 = -\lambda$$

$$x(t) = c_1 e^{-\lambda t} + c_2 t e^{-\lambda t}$$

Case III: $\lambda^2 - k^2 < 0$ (underdamped)

$$n_1 = -\lambda + i\sqrt{k^2 - \lambda^2} \qquad n_2 = -\lambda - i\sqrt{k^2 - \lambda^2}$$

$$x(t) = e^{-\lambda t}(c_1 cos(\sqrt{k^2 - \lambda^2} t) + c_2 sin(\sqrt{k^2 - \lambda^2} t)$$

The particular solution, $x_p(t) = \frac{k^2-w^2}{(k^2-w^2)+4w^2\lambda^2}\cos(wt) + \frac{2\lambda w}{(k^2-w^2)+4w^2\lambda^2}\sin(wt)$ is used to model any external forces that are acting upon the mass system. When modeling the mass systems, we must consider external forces in order for the model to be accurate in the real world. These external forces come in many forms. When modeling the design of shock absorbers, we can consider speed bumps or suddenly breaking as an external force. When designing a paraplegic door, we consider the force exerted on the door when one opens or closes it, when they don't use the button. In a more complex case when we examine the suspension bridge, this particular model does simplify the scenario since we are only considering one external force in a system which exhibits many external forces. However, if we choose to consider one force, this external force could be something such as wind velocity which can cause the bridge to swing at higher amplitudes.

We do take some liberties in just considering the case where our homogeneous equation is equal to $\cos(wt)$. This is representative of a *possible* external force meaning there could be some error in the modeling. However, through further research more accurate models can be formed and catered to specific applications.

Some important points to consider are the type of damping that occurs in each equation. The case of underdamping relates most closely to objects under harmonic motion. This is due to the oscillatory nature of the sine and cosine functions. Damping happens due to the $e^{-\lambda t}$, which behaves as a strictly decreasing amplitude for the characteristic equation. The value gets smaller and smaller as time increases, which implies the maximum and minimum values of the characteristic equation approach 0 as t approaches ∞. The particular solution for all cases will still be in motion but the damping force applied to the object is strong enough to keep the object close to equilibrium.

Below are some images of the graphs of each damping case. The independent variable is the time and the displacement is our dependent variable.

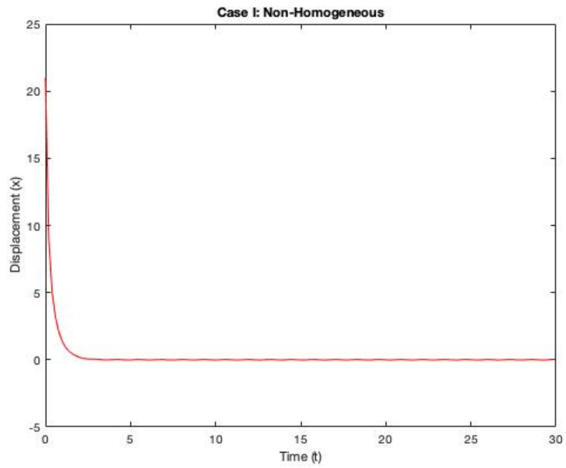

Figure 4. Graphical images of the non-homogeneous version of overdamped equation.

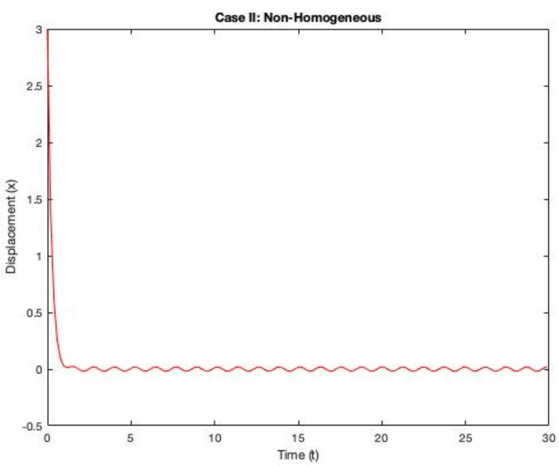

Figure 5. Graphical images of the non-homogeneous version of critically damped equation.

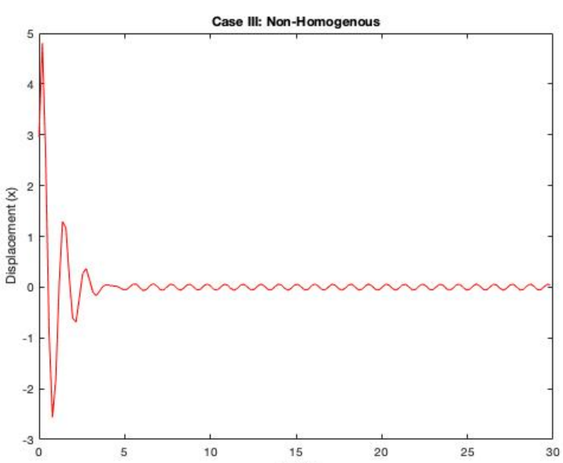

Figure 6. Graphical images of both the homogeneous and non-homogeneous versions of underdamped equation.

As the graphs show, the damping force has a large impact on the expression of these equations. When the damping constant is larger than the spring constant, the graph does not oscillate basically at all, attempting to return to its equilibrium state as quickly as possible. In the case of critically damped, the situation is very similar, except the damping constant and spring constant are equal to each other. This can be reflected in the graph, following suit after the overdamped equation, but it does not reach equilibrium as quickly. In the case of the non-homogeneous case, the system continues to oscillate, but with a much lower amplitude. In our final case, the underdamped situation, the graph crosses the x-axis many times, further reinforcing the notion that the damping constant is smaller than the spring constant. This causes the system to oscillate much more than the other two cases. As time progresses, the amplitudes of the graphs decrease dramatically in an attempt to stay at equilibrium. Overall, these graphs are important for a deeper understanding of the meaning behind each equation.

Conclusion

In conclusion, through this paper we have been able to review the importance of mathematics and mathematical modeling. Having discussed differential equations and simple harmonic motion equations, we were able to understand the causes and effects of external forces and their oscillating object that occur in real life application such as: shock absorbers, guitar strings, and springs. With this paper students will be able to have a thorough grasp of these equations which creates a basis for the understanding of its applications. As it is seen that the simple harmonic equations configured through Hooke's Law and Newton's Second Law of physics. While the study of these ordinary differential equations may seem simple, their applications are found in other STEM fields. These fields can include, but are limited to: Chemistry, Engineering, and Physics.

Acknowledgements

I would like to thank the California State University, Stanislaus Honors Program and McNair Scholars program for providing me with the necessary support to complete this research. I would like to express my gratitude to Dr. Ellen Bell, Mrs. Araceli Garcia, Dr. Andrew Dorsey for their guidance and assistance throughout this research. I would also like to give a special recognition to Dr. JungHa An. Without her guidance and support, this project would not have been made possible.

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Appendix:

Coefficients/ Terms	Meaning
$x(t)$	This term represents an object's displacement over time. The way we can interpret this term is how far an object is oscillating from its equilibrium (starting) position
$x''(t)$	The acceleration of an object over time. This would be the acceleration of each oscillation of an object.
k^2	The k^2coefficient is used to help find the period constant for the characteristic equation of our solutions. k^2represents the square of the period constant. The characteristic equation is the part of the solution $x(t)$ found when calculating the homogeneous solution. Another characteristic of $k^2$is $k^2 = \frac{s}{m}$, where m is the mass and s is the spring constant. These are important because they help determine the coefficient in the harmonic motion equation.
w	The period constant of the function. This coefficient is used to find the period, $T = \frac{2\pi}{w}$, where T is the time in seconds for oscillations of some external force. A period constant, k, is also present for the object's oscillations. We can use this formula for period because the oscillations create a sinusoidal wave. This coefficient is also used for finding the Frequency, which is $F = \frac{1}{T} = \frac{w}{2\pi}$, the amount of oscillations per second.
$cos(wt)$	This function in (2) and (3) is representative of some sort of external force on the system that causes it to go out of equilibrium. This could be a driving force that causes greater oscillations of the object.
$k^2x(t)$	This term is said to describe a part of the restoring force, which is a force that gives rise to an object in equilibrium. This term is composed by the product of the square of an object's period constant and its displacement. This term follows Hooke's Law.
2λ	2λ is referred to as the damping force of the mass system. This force allows the mass system to return to equilibrium and makes it a more applicational problem rather than a theoretical equation. The coefficient can be broken down as $2\lambda = \frac{\beta}{m}$, where $\beta$is the damping force and m is the mass of the object.
β	β is considered the positive damping constant of the equation. In most cases, the $\beta$is negative because it moves in the opposite direction of motion.

Table 1: Explanation and Importance of Terms and Coefficients.