Math 1410 Name:____________________________

Worksheet #11: Section 3.2

In the previous section, we defined a new object called the derivative of f at $x = a$. We'll restate this definition here.

Definition 1. The derivative of f at $x = a$, called $f'(a)$ and read as "f prime of a," is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \text{ if this limit exists.}$$

Computing derivatives at arbitrary values of x is quite useful. In fact, by doing this, we end up looking at derivatives as functions of x. We do this by replacing a with a variable x.

Definition 2. The derivative of a function f, denoted by f', is given by

$$f'(x) =$$

Problem 1. Let's compute some derivatives. Find the derivative of $f(x) = x^2 + 3x$.

Problem 2. Find the derivative of $f(x) = \dfrac{4}{x+3}$.

Notice that the domain of f' can be no larger than the domain of f since the definition of $f'(x)$ requires using $f(x)$. If f is undefined at a given value of x, then it is impossible to compute f' at that value of x. If you look at the last two examples, the domains of f and f' are the same. On the other hand, in the last worksheet, we found that if $f(x) = \sqrt{4-x}$, then by replacing a with x, we have $f'(x) = \dfrac{-1}{2\sqrt{4-x}}$. In this case. . .

The domain of f is

The domain of f' is

There is a lot of terminology involving the derivative and the process of computing a derivative. We'll go through much of it here, starting with some definitions.

Definition 3. The process of taking a derivative is also known as

Definition 4. A function f is <u>differentiable at a</u> if

Definition 5. A function f is <u>differentiable on an open interval</u> if

Problem 3. Using your earlier work, where is $f(x) = \sqrt{4-x}$ differentiable?

In addition to the notation $f'(x)$, there are a bunch of other notations that are commonly used. If our function is written as $y = f(x)$, then the following symbols all represent the derivative of f:

Moreover, the following notations represent the value of the derivative of $f(x)$ at $x = a$:

Problem 4. Where is $f(x) = |x - 1|$ differentiable?

Continuity and differentiability are wonderful properties to have. In fact they are intertwined with each other. The following theorem cements this.

Theorem 1. If f is differentiable at $x = a$, then

Earlier, we saw that $f(x) = \dfrac{4}{x+3}$ is differentiable on $(-\infty, -3) \cup (-3, \infty)$. By the theorem above, we can conclude

Keep in mind that THE CONVERSE IF FALSE! A function can be continuous at $x = a$ but not differentiable there. The last example shows that while $f(x) = |x - 1|$ is continuous everywhere, $f'(1)$ does not exist.

So, how can a function fail to be differentiable at $x = a$?

1. Discontinuity:

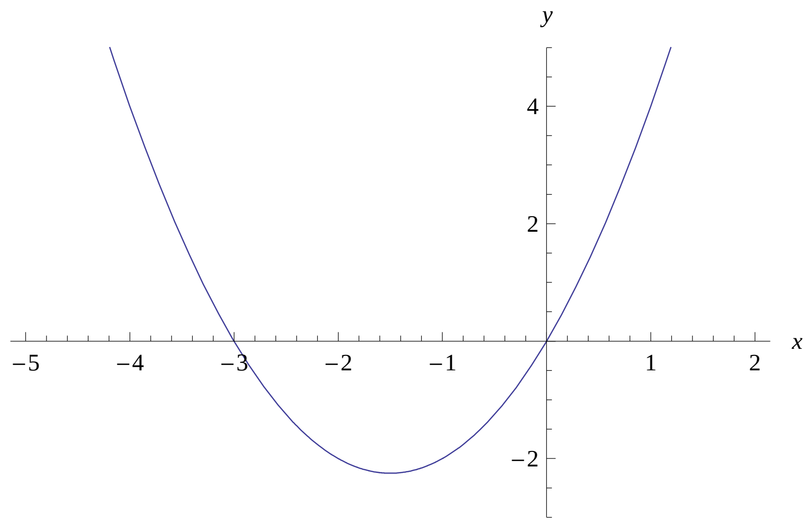
2. Corner:

3. Cusp:

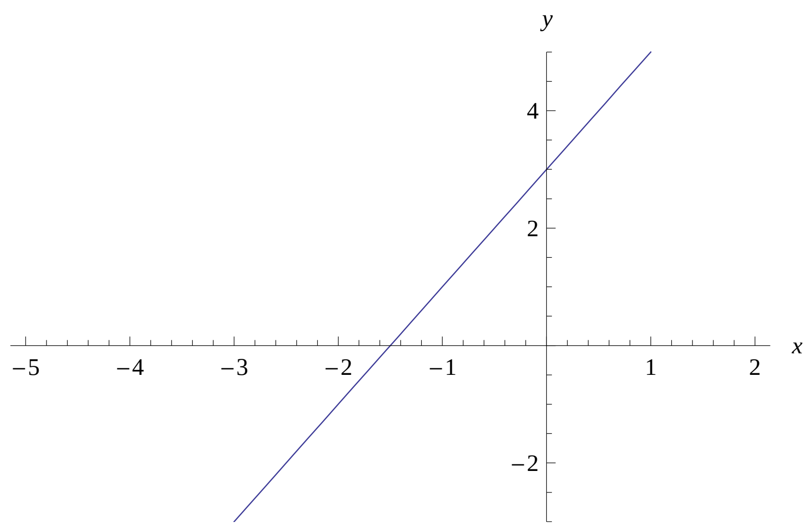
4. Vertical tangent line:

Now we shall take some time to learn how to draw a graph of $f'(x)$ if we have a graph of $f(x)$. In the first problem, we found that if $f(x) = x^2 + 3x$, then $f'(x) = 2x + 3$.

$y=x^2+3x$



$y=2x+3$



Based on this work, we can draw the following conclusions:

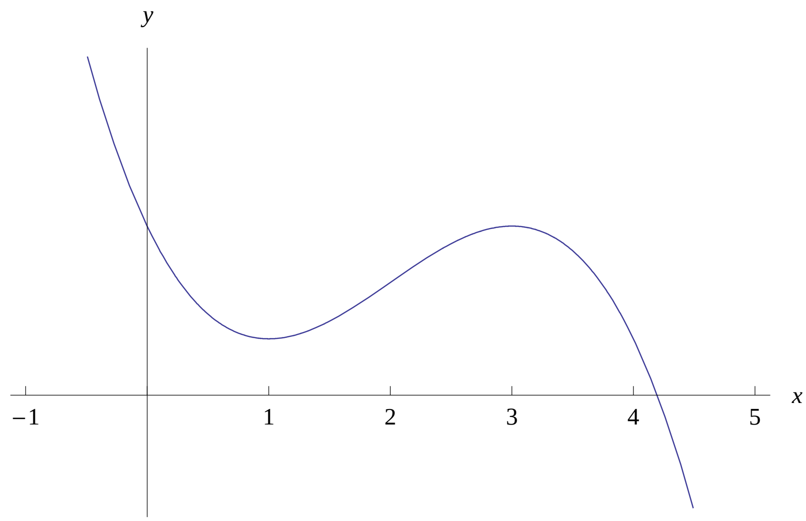
1. If the slope of the tangent line to $f(x)$ at $x = a$ is zero, then the graph of $f'(x)$

2. If the slope of the tangent line to $f(x)$ at $x = a$ is positive, then the graph of $f'(x)$

3. If the slope of the tangent line to $f(x)$ at $x = a$ is negative, then the graph of $f'(x)$

4. As the steepness of the tangent line increases, the graph of $f'(x)$ moves further away from the x-axis.

Problem 5. You are given the following graph of f. Draw f'.



In our computations of derivatives, we started with a function $f(x)$ and computed $f'(x)$. But $f'(x)$ is a function on its own, which means it may have a derivative of its own. So, we can compute $(f')'$, which is often written as f'' and is called the second derivative of f.

Definition 6. The second derivative of a function f, denoted by f'', is given by

$$f''(x) =$$

Problem 6. In the first problem, we saw if $f(x) = x^2 + 3x$, then $f'(x) = 2x + 3$. Compute $f''(x)$.

Just as with the first derivative, the second derivative has multiple notations. Given a function $y = f(x)$, the following are notations for the second derivative of f.

Similarly, we can define the third derivative as

Notations for the third derivative include

Higher derivatives are defined similarly.

Keep in mind that the derivative of f represents the instantaneous rate of change of f. This means the second derivative of f is the rate of change of the rate of change of f. In particular, f'' describes how much the rate of change is changing.

Higher derivatives do have some physical interpretations. Suppose that $s(t)$ is the position of an object at time t. We know that $v(t) = s'(t)$ is the instantaneous velocity of the object at time t. Keeping in mind that derivatives are instantaneous rates of change, we can say that $v'(t) = s''(t)$ is the instantaneous rate of change of velocity. What is this more commonly called? (Usually this is denoted as $a(t)$.)

If we then take the third derivative of $s(t)$, which gives us $s'''(t) = v''(t) = a'(t)$, then we have the instantaneous rate of change of $a(t)$. You've certainly felt this before if you hit the brakes or accelerator really hard while driving. What is this called? (Usually this is denoted as $j(t)$.)

We now have the following set-up:

$s(t) =$ position

$v(t) = s'(t) =$ velocity

$a(t) = v'(t) = s''(t) =$

$j(t) = a'(t) = v''(t) = s'''(t) =$