Real and Complex Dimensions of Fractal Strings

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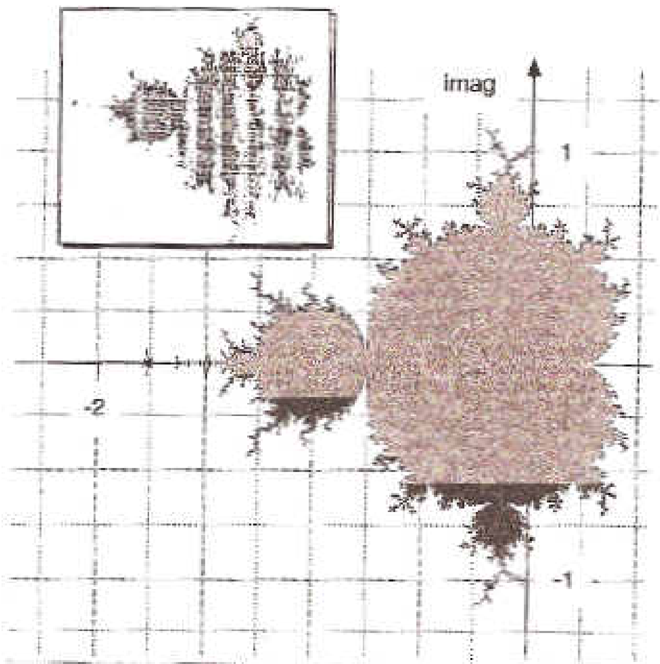
The Mandelbrot Set

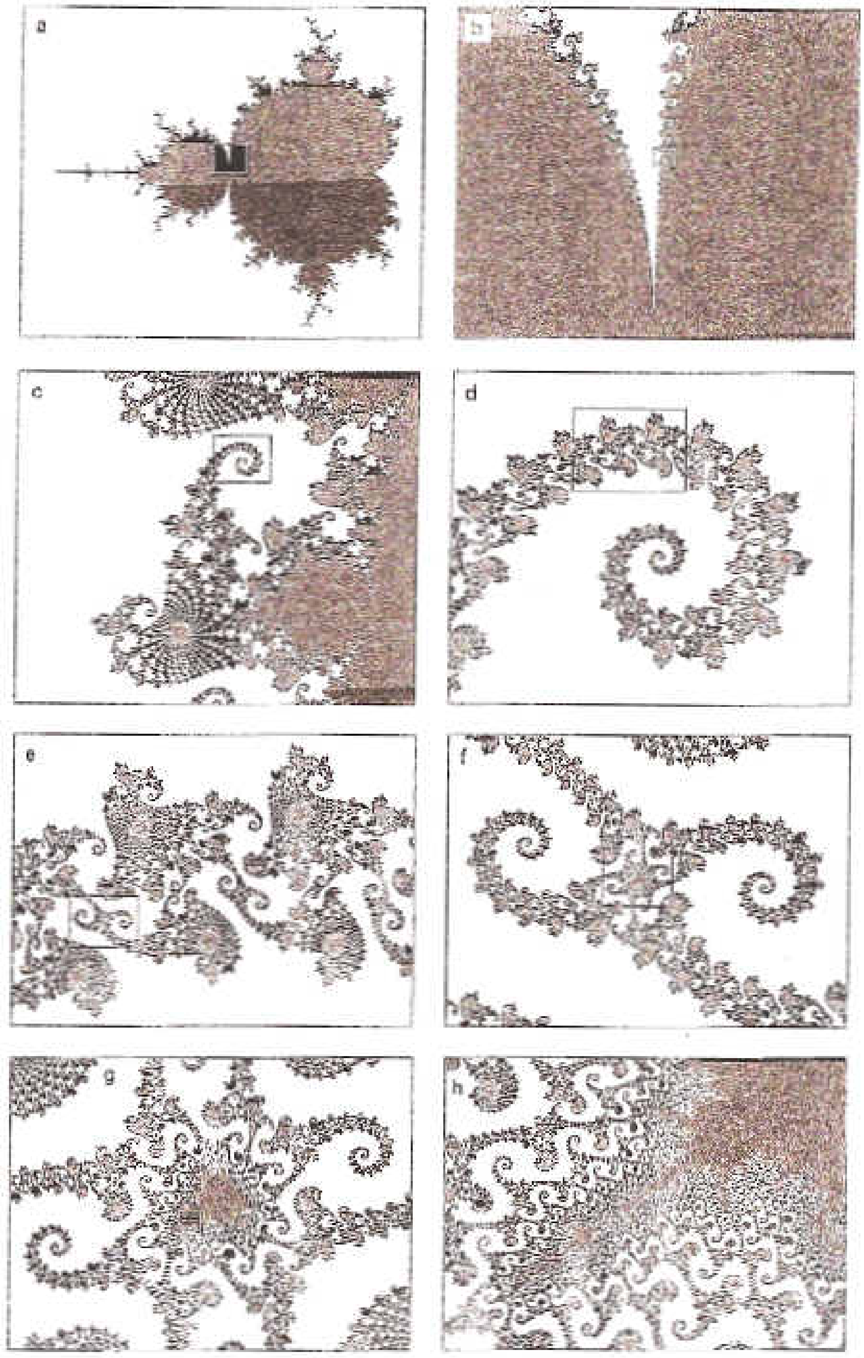
$$M = \{c \in \mathbb{C} \mid 0 \to c \to c^2 + c \to \dots \text{ remains bounded}\}$$



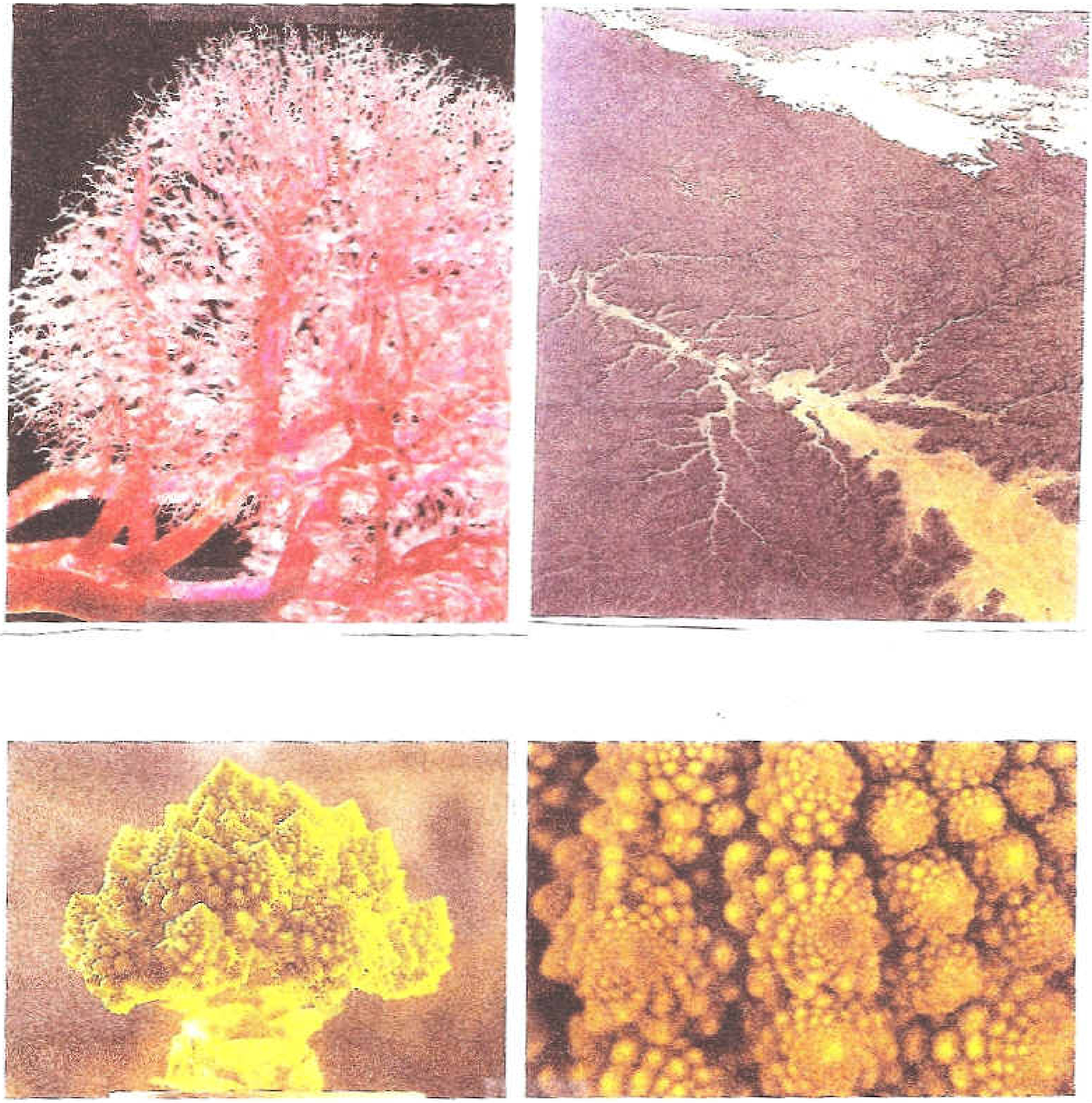
See 4, p. 786

Zooming into the Mandelbrot set M.

See (4) p. 798



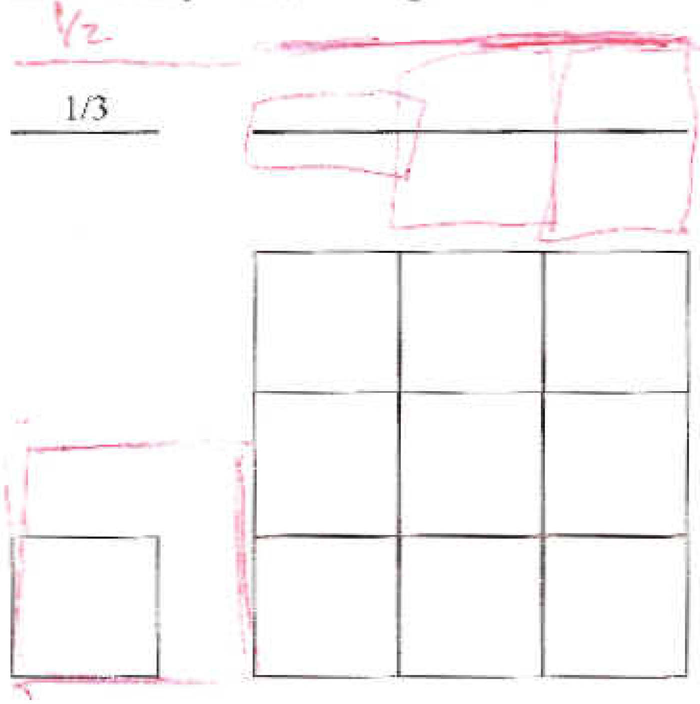
Some real-life fractals. See 4, p. 130-131



How many boxes of side length 1/3 does it take to cover each of these objects?

How do they scale, from right to left?

$(1/2)^{-1} = 2$



\# boxes to cover:

$(1/3)^{-1} = 3$

$\left(\frac{1}{2}\right)^{-2} = 4$

$(1/3)^{-2} = 9$

The dimensions $D = 1$ and 2 for each type of figure above satisfy the following equivalent equations:

$$N_\varepsilon = \varepsilon^{-D} \Leftrightarrow D = \frac{\log N_\varepsilon}{-\log \varepsilon}$$

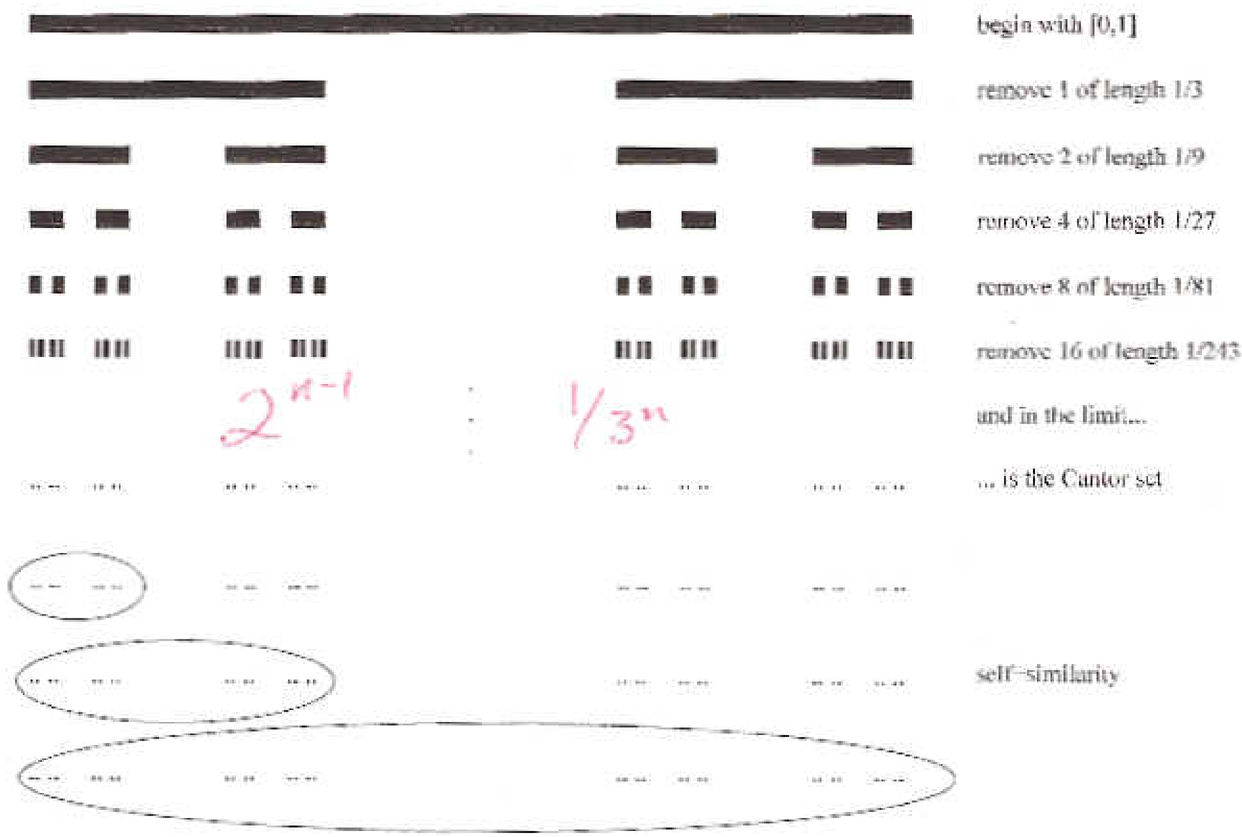
where ε is the size of the side length of the boxes used to cover the objects and N_ε is the minimum number of boxes required to cover them.

The method of counting boxes allows one to define a notion of dimension that assigns a real value (as opposed to simply an integer value) to a subset of Euclidean space.

Definition 1. *The* <u>*box dimension*</u> $\dim_B$ *of a bounded subset* F *of* $\mathbb{R}^m$ *is given by the following limit (when it exists):*

$$\dim_B(F) = \lim_{\varepsilon \to 0^+} \frac{\log N_\varepsilon(F)}{-\log \varepsilon},$$

where $N_\varepsilon(F)$ *is the smallest number of "cubes" with side length* ε *that cover* F.



The Cantor set is constructed by removing 2^{n-1} open intervals of length $1/3^n$ for every $n \in \mathbb{N}$ from the unit interval $[0, 1]$.

Definition 2. *A fractal string Ω is a bounded open subset of the real line. The collection of lengths ℓ_j of the disjoint intervals is denoted $\mathcal{L}$.*

Theorem 3. *If a fractal string Ω in $[0,1]$ is of total length 1 and has an infinite number of lengths in its sequence $\mathcal{L}$, then*

$$\dim_B(\partial\Omega) = \inf\left\{\sigma \in \mathbb{R} \mid \sum_{j=1}^{\infty} \ell_j^{\sigma} < \infty\right\},$$

where $\partial\Omega = [0,1] \setminus \Omega$.

Definition 4. *The geometric zeta function of a fractal string Ω with lengths $\mathcal{L}$ is*

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} \ell_j^{s} = \sum_{n=1}^{\infty} m_n l_n^{s},$$

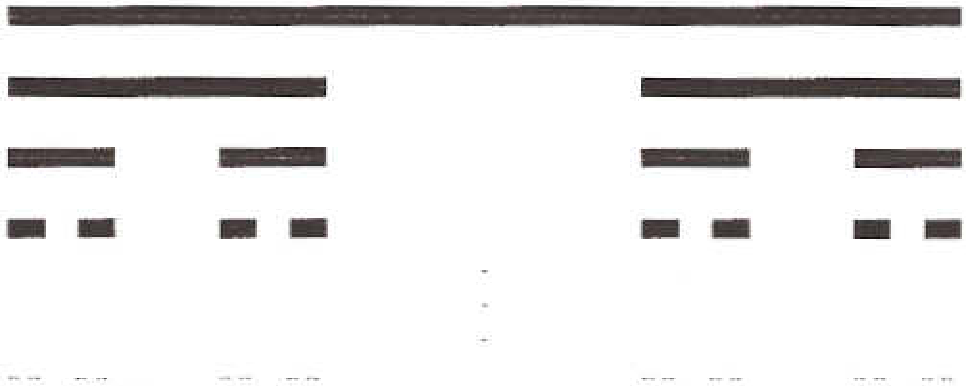
where $\mathrm{Re}\, s > \dim_B(\partial\Omega)$.

$\sum_{j=1} l_j^{s}$

Definition 5. *The set of complex dimensions of a fractal string Ω with lengths $\mathcal{L}$ is*

$$\mathcal{D}_{\mathcal{L}}(W) = \{\omega \in W \mid \zeta_{\mathcal{L}} \text{ has a pole at } \omega\}.$$

where W is a certain region in the complex plane.



The Cantor set exhibits the following properties:

- It has length 0. $\sum \frac{2^{n-1}}{3^n} = 1$
- It is a closed and perfect subset of $[0,1]$ (every point is a limit point).
- It is uncountable.
- It is self-similar.
- It has box-counting dimension equal to $\cancel{\log_2 3}$. $0 < \log_3 2 < 1$

Note that when boxes of size $\varepsilon = 3^{-n} = 1/3^n$ are used, a minimum number $N_\varepsilon(F) = 2^n$ boxes are required to cover the set. Thus,

$$\lim_{\varepsilon \to 0^+} \frac{\log N_\varepsilon(F)}{-\log \varepsilon} = \lim_{n \to \infty} \frac{\log(2^n)}{-\log(3^{-n})} = \log_3 2$$

Definition 6. *The regularity $A(U)$ of a (Borel) measure μ on a subset $U \subset [0,1]$ with range in $[0,\infty]$ is*

$$A(U) = \frac{\log \mu(U)}{\log |U|},$$

where $|\cdot| = \lambda(\cdot)$ is the Lebesgue measure on $[0,1]$.

Equivalently, $A(U)$ is the exponent α that satisfies

$$|U|^\alpha = \mu(U).$$

Definition 7. *For a measure μ and ordered family of partitions $\mathfrak{P}$, the partition zeta function with regularity α is*

$$\zeta^\mu_{\mathfrak{P}}(\alpha, s) = \sum_{n=1}^{\infty} \sum_{k=1}^{p_n} \delta_\alpha(P^k_n)|P^k_n|^s,$$

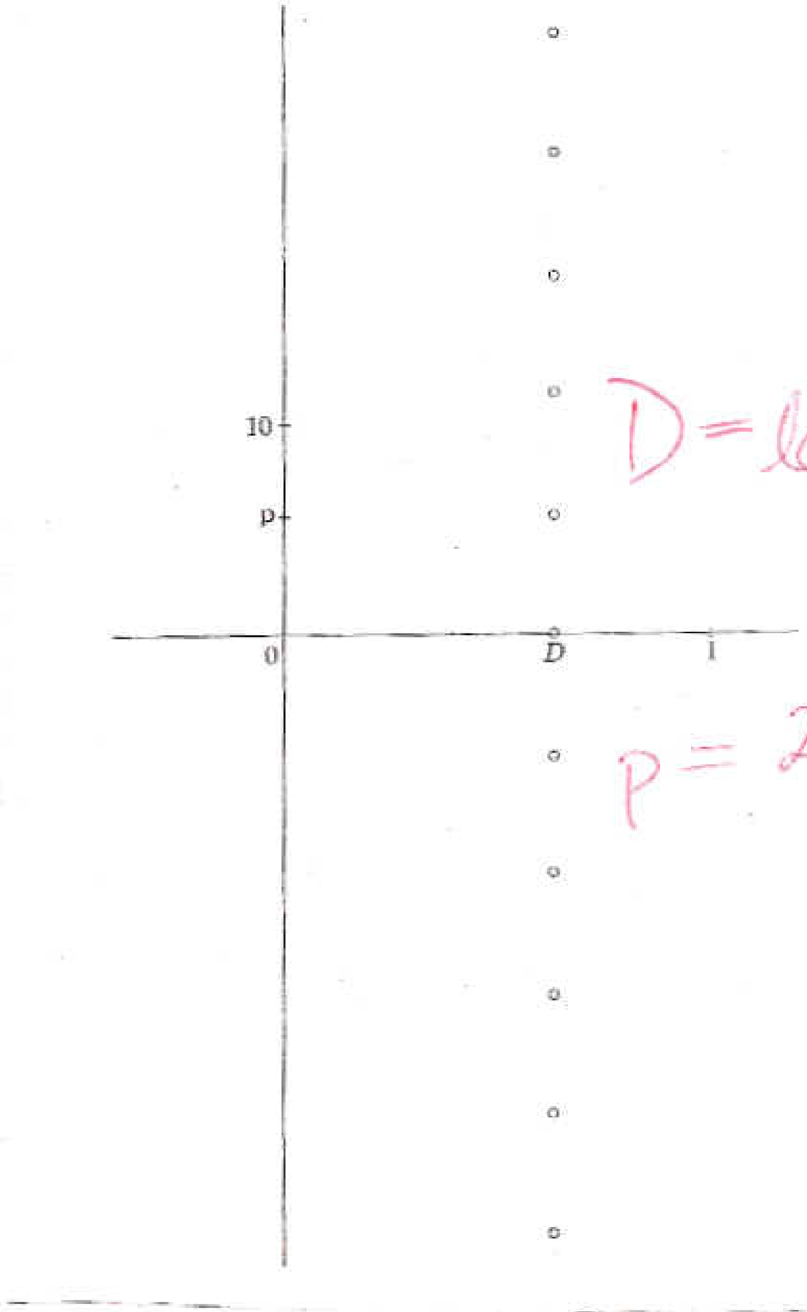
where $\delta_\alpha(P)$ equals 1 if $A(P) = \alpha$ and equals 0 otherwise, p_n is the number of intervals in the partition $\mathcal{P}_n$, and $\mathrm{Re}\, s$ is large enough.

For the measure ν and the family $\mathfrak{P}$ of partitions given by the open and closed intervals in the construction of the Cantor set, the regularity values are

$$\alpha = \alpha(k_1, k_2) = \frac{\log\left(2^{nk_1}/3^{nk_2}\right)}{\log\left(1/3^{nk_2}\right)} = 1 - \frac{k_1}{k_2}\log_3 2,$$

where k_1 and k_2 are relatively prime non-negative integers such that $k_1 < k_2$.

The complex dimensions of the Cantor String.

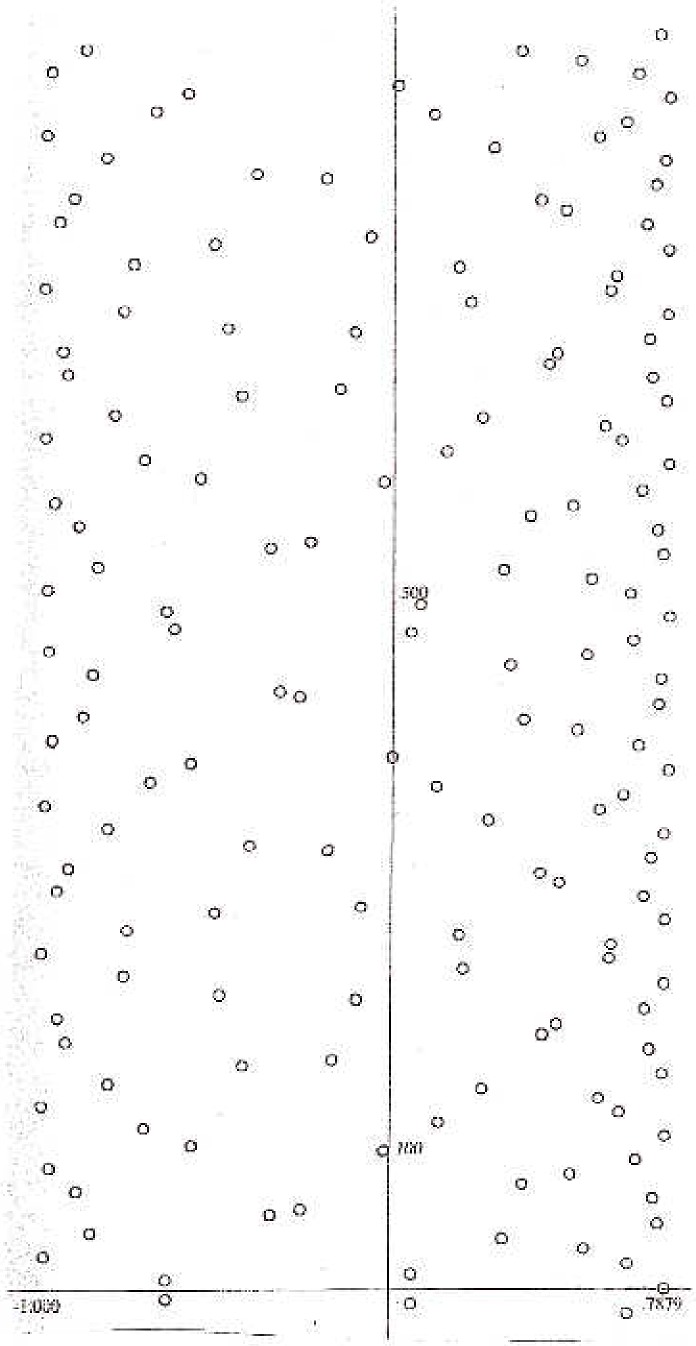


$D = \log_3 2$

$p = 2\pi / \log 3$

See (3), p. 42

The complex dimensions of another fractal string.



1/2 1/3
1/6

See (3), p. 49

Nifty things one can do with fractals strings, zeta functions, and complex dimensions:

- Find the box-counting dimension of the complements of fractal strings (Theorem 3).
- Find the volume of the inner ϵ-neighborhood of the boundary of certain fractals.
- Investigate properties of fractal strings and multifractal measures.
- Show, in a new way, that the Critical Zeros of the Riemann Zeta Function do not lie in vertical arithmetic progression.
- Reformulate the Riemann Hypothesis as an inverse spectral problem.

$$\zeta_{\nu}(s) = \underline{\zeta(s)}\, \zeta_{\mathcal{L}}(s)$$

For the Cantor String, the geometric zeta function is

$$\zeta_{\mathcal{L}}(s) = \zeta_{CS}(s) = \sum_{n=1}^{\infty} 2^{n-1}3^{-ns} = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}}.$$

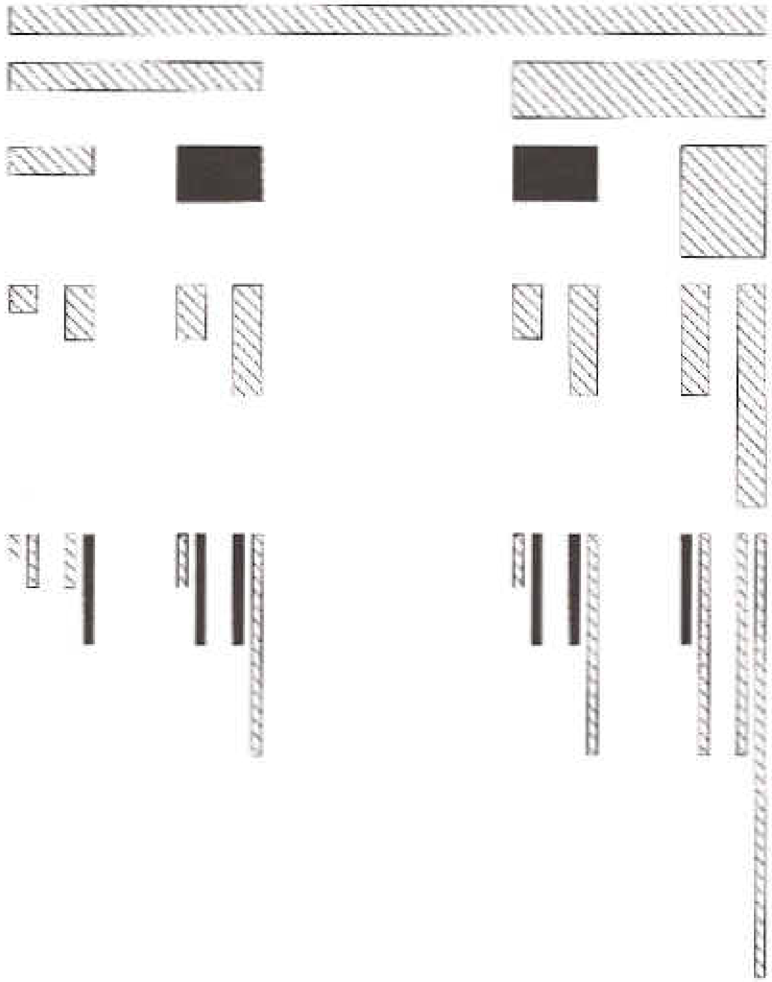
$$2^{n-1}3^{-ns}$$

Upon meromorphic continuation, we see that the last equation above holds for all $s \in \mathbb{C}$, hence

$$\mathcal{D}_{\mathcal{L}} = \mathcal{D}_{CS} = \left\{ \log_3 2 + \frac{2im\pi}{\log 3} \mid m \in \mathbb{Z} \right\}.$$

1
1 1
1 2 1
1 3 3 1

The first few stages in the construction of a mass distribution ν on the Cantor set. At each stage, mass is split from the previous stage in ratios of 1/3 on the left and 2/3 on the right.

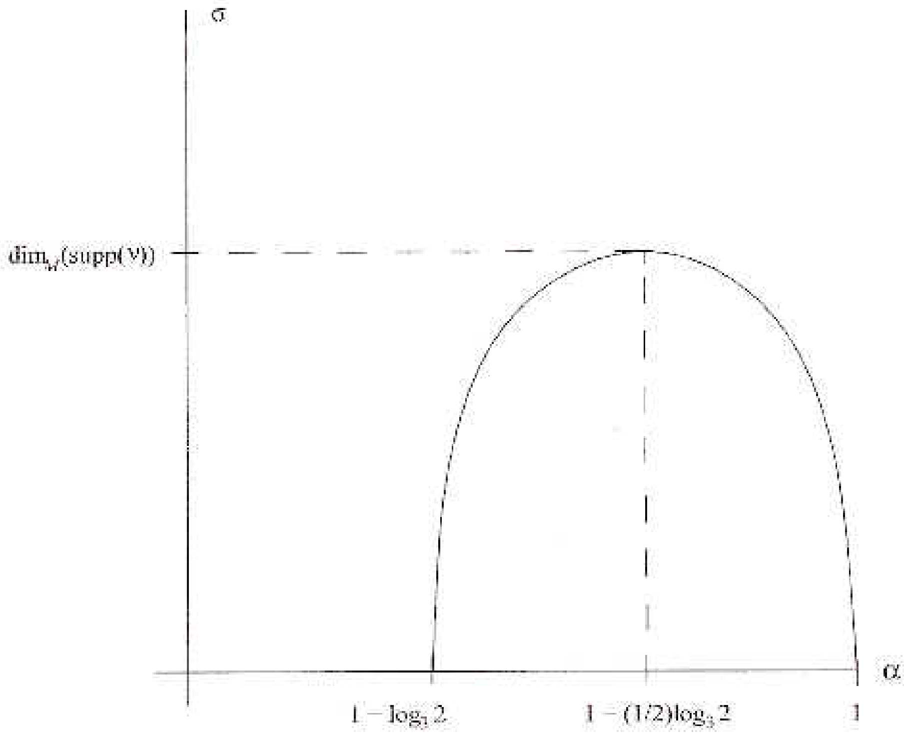


$$\alpha(1,2) = \frac{\log\left(2^n/3^{2n}\right)}{\log\left(1/3^{2n}\right)} = 1 - \frac{1}{2}\log_3 2$$

Construction of the partition zeta function $\zeta_{\mathfrak{P}}^{\nu}(\alpha(1,2), s)$. The solid black blocks correspond to the closed intervals with regularity $\alpha(1,2)$. In general,

$$\zeta_{\mathfrak{P}}^{\nu}(\alpha(k_1, k_2), s) = \sum_{n=1}^{\infty} \binom{nk_2}{nk_1} 3^{-nk_2 s},$$

for $\operatorname{Re} s$ large enough.



The abscissa of convergence function σ for the measure ν has the form

$$\begin{aligned}\sigma(\alpha) &= \frac{(\alpha-1)}{\log_3 2}\cdot\log_3\left(\frac{-(\alpha-1)}{\log_3 2}\right)\\ &\quad -\left(1+\frac{(\alpha-1)}{\log_3 2}\right)\cdot\log_3\left(1+\frac{(\alpha-1)}{\log_3 2}\right).\end{aligned}$$

The maximum of σ is attained at $\alpha = \alpha(1,2) = 1-(1/2)\log_3 2$ and this value coincides with the box dimension of the support of the measure ν. That is,

$$\begin{aligned}\dim_B(supp(\nu)) &= \max\{\ \sigma(\alpha)\mid \alpha=\alpha(k_1,k_2)\}\\ &= \log_3 2.\end{aligned}$$

REFERENCES

1 • K. Falconer, *Fractal Geometry (Mathematical Foundations and Applications)*, 2nd ed., John Wiley, Chichester, 2003.

2 • M. L. Lapidus, J. Lévy Véhel, J. A. Rock, Fractal strings and multifractal zeta functions, submitted to *Adv. Math.* October 2006. (Also, e-print arXiv:math-ph/0610015, 2006.)

3 • M. L. Lapidus and M. van Frankenhuijsen, *Fractal Geometry, Complex Dimensions and Zeta Functions*, Springer 2006.

4 • H-O. Peitgen, H. Jürgens and D. Saupe, *Chaos and Fractals*, 2nd ed., Springer 2004.

5 • J. A. Rock, *Zeta Functions, Complex Dimensions of Fractal Strings, and Multifractal Analysis of Mass Distributions*, thesis, June 2007.

6 • Website for Mandelbrot set applet:
http://math.hws.edu/xJava/MB/

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This handout provides a synopsis for my talk, which consists of a small sample of some well-known notions in Fractal Geometry.

What is a fractal? There is no universally agreed-upon definition of fractal, but generically, a fractal is a physical or mathematical object that exhibits interesting structure on all scales.

Some physical examples include human lungs, arterial and venous systems of kidneys, broccoli romanesco, mountains, shorelines, ferns, clouds, turbulence, etc.

Some mathematical examples include the famous Mandelbrot set and the Cantor set.

Multifractals are objects which exhibit fractal structure in a variety of ways (and are sometimes parameterized by real values).

Self-similarity is a property shared by both the Cantor set and the broccoli romanesco: Both have subsets at ever-decreasing scales that are copies of the full-scale object.

Box (a.k.a. Minkowski) dimension is a generalization of integer dimension which assigns real-numbered values for many sets in Euclidean space.

Complex dimensions generalize the box dimension by assigning a family of complex numbers to certain subsets of the real line $\mathbb{R}$.

Fractal Strings are bounded, open subsets of $\mathbb{R}$. They are simple mathematical objects whose boundaries (in the right setting) have fractal structure.

Handout 1

The Cantor set is the complement in $[0, 1]$ of a special fractal string called the Cantor String. The Cantor set exhibits the following properties:

- It has length 0.
- It is a closed and perfect subset of $[0, 1]$ (every point is a limit point).
- It is uncountable.
- It is self-similar.
- It has box-counting dimension equal to $\log_3 2$.

The Geometric Zeta Function of a fractal string Ω with lengths $\mathcal{L}$ is

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The Complex Dimensions of a fractal string Ω with lengths $\mathcal{L}$ is

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Handout 2