

# A FRACTAL APPROACH TO THE RIEMANN HYPOTHESIS

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$$\{s \in \mathbb{C} \mid \zeta(s) = 0, 0 < \operatorname{Re}(s) < 1\} \subset \{s \in \mathbb{C} \mid \operatorname{Re}(s) = 1/2\}$$

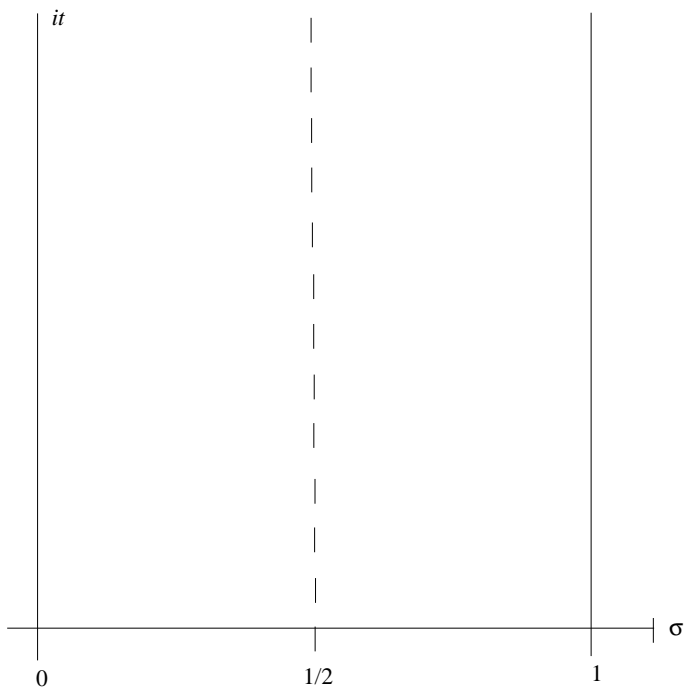
## What is it?

It is what it is... just kidding.

First, the Riemann zeta function. It has many forms, including

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where  $s = \sigma + it \in \mathbb{C}$ . Naturally defined for  $\operatorname{Re}(s) > 1$ , it has a meromorphic continuation to  $\mathbb{C}$ .

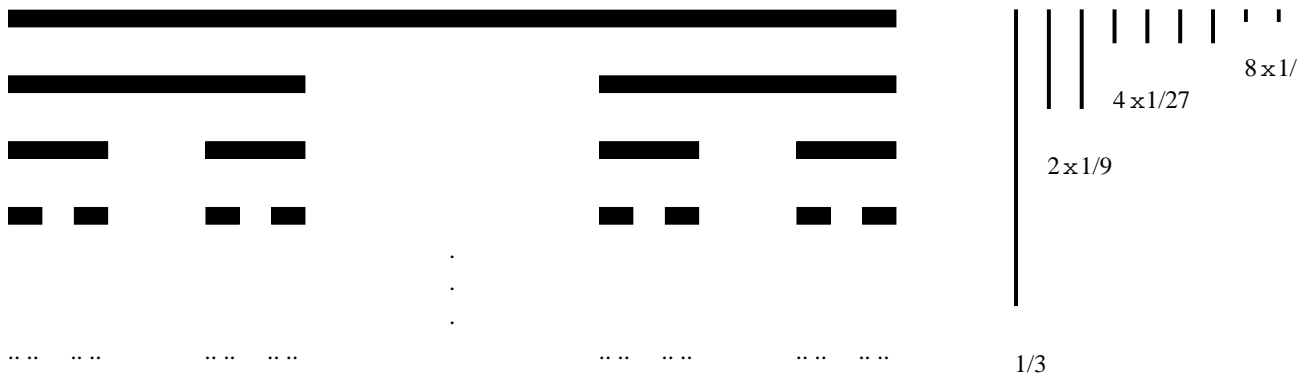


The Riemann hypothesis states that all of the nontrivial zeros of the Riemann zeta function have real part  $1/2$ .

$$\{s \in \mathbb{C} \mid \zeta(s) = 0, 0 < \operatorname{Re}(s) < 1\} \subset \{s \in \mathbb{C} \mid \operatorname{Re}(s) = 1/2\}$$

## A little history.

- 1859 – Riemann first mentions the conjecture in his paper *On the Number of Primes Less Than a Given Magnitude*.
- 1896 – Hadamard and de la Vallée-Poussin independently show that all nontrivial zeros lie in the interior of the Critical Strip  $0 < \text{Re}(s) < 1$ . This leads to the first complete proofs of the prime number theorem.
- 1900 – Hilbert includes the Riemann hypothesis in his list of 23 unsolved problems.
- 1901 – Von Koch proves that the Riemann hypothesis is equivalent to a stronger version of the prime number theorem.
- 1914 – Hardy proves that there are an infinite number of zeros on the critical line  $\text{Re}(s) = 1/2$ .
- 1995 – Lapidus and Maier provide a restatement of the Riemann hypothesis in terms of the geometric and spectral oscillations of fractal strings, stemming from the question “*Can one hear if a fractal string is Minkowski measurable?*”
- 1997 – Lapidus and van Frankenhuysen show that the Riemann zeta function does not have an infinite sequence of critical zeros in arithmetic progression.



### The context.

We will look at the Riemann zeta function as it arises in the study of the geometry and spectra of fractal strings.

**Definition 1.** A *fractal string*  $\Omega$  is a bounded open subset of the real line. The collection of lengths  $\ell_j$  of the disjoint intervals is denoted  $\mathcal{L}$ .

**Theorem 2.** *If a fractal string  $\Omega$  in  $[0, 1]$  is of total length 1 and has an infinite number of lengths in its sequence  $\mathcal{L}$ , then*

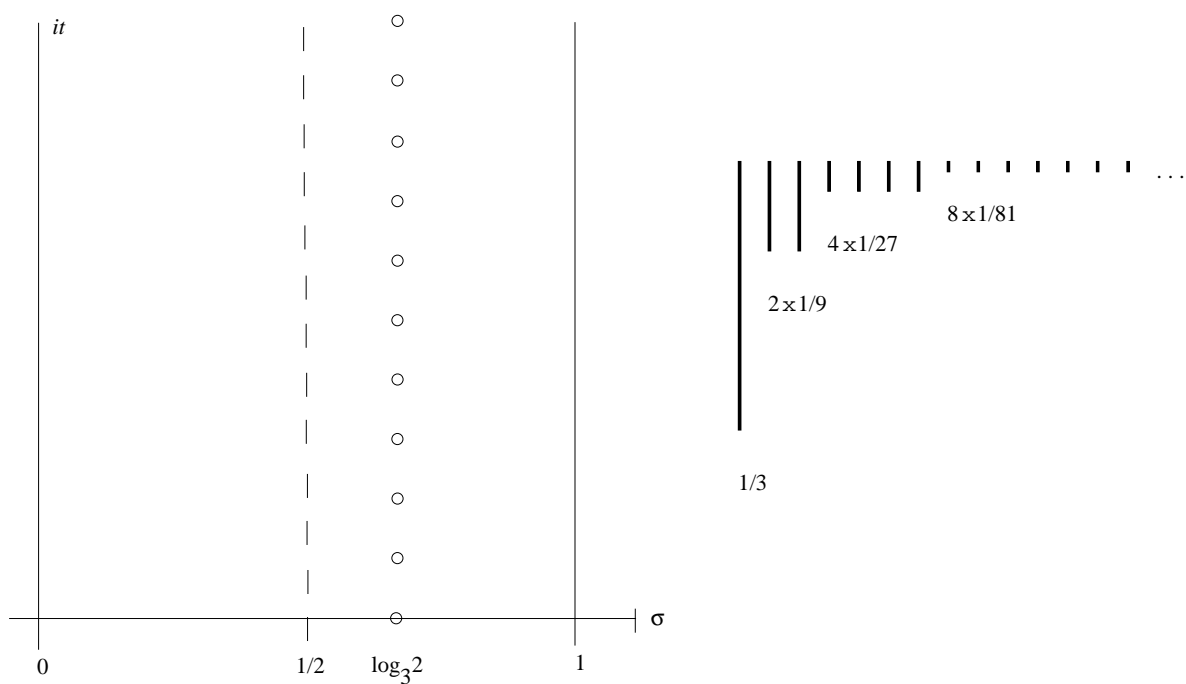
$$\dim_B(\partial\Omega) = \inf \left\{ \sigma \in \mathbb{R} \mid \sum_{j=1}^{\infty} \ell_j^\sigma < \infty \right\},$$

where  $\partial\Omega = [0, 1] \setminus \Omega$ .

**Definition 3.** *The geometric zeta function of a fractal string  $\Omega$  with lengths  $\mathcal{L}$  is*

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} \ell_j^s = \sum_{n=1}^{\infty} m_n l_n^s,$$

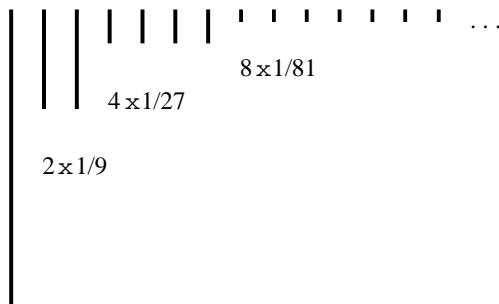
where  $\operatorname{Re}(s) > \dim_B(\partial\Omega)$ .



**Definition 4.** *The set of complex dimensions of a fractal string  $\Omega$  with lengths  $\mathcal{L}$  is*

$$\mathcal{D}_{\mathcal{L}}(W) = \{\omega \in W \mid \zeta_{\mathcal{L}} \text{ has a pole at } \omega\}.$$

*where  $W$  is a certain region in the complex plane.*



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## Frequencies and the Spectral Zeta Function.

Given a fractal string  $\Omega$  with lengths  $\mathcal{L}$ , we can listen to its sound. That is, consider the Dirichlet Laplacian  $\Delta = -d^2/dx^2$ . An eigenvalue  $\lambda$  of  $\Delta$  corresponds to the normalized frequency  $f = \sqrt{\lambda}/\pi$  of the fractal string. The frequencies of an interval of length  $l$  are  $k \cdot l^{-1}$ . Thus the frequencies of  $\mathcal{L}$  are

$$f = k \cdot \ell_j^{-1}$$

where  $k, j \in \mathbb{N}$ . The total multiplicity of the frequency  $f$  is

$$w_f^{(\nu)} = \sum_{j: f \cdot \ell_j \in \mathbb{N}^*} 1 = \sum_{l: f \cdot l \in \mathbb{N}^*} w_l = w_{1/f} + w_{2/f} + \dots$$

**Definition 5.** The spectral zeta function of  $\mathcal{L}$  is

$$\zeta_\nu(s) = \sum_{k,j=1}^{\infty} (k \cdot \ell_j^{-1})^{-s} = \sum_f w_f^{(\nu)} f^{-s}$$

where  $f$  runs through the distinct frequencies of  $\mathcal{L}$ .

The Riemann zeta function connects the geometric and spectral zeta functions of a fractal string with lengths  $\mathcal{L}$ .

**Theorem 6.** The spectral zeta function of  $\mathcal{L}$  is given by

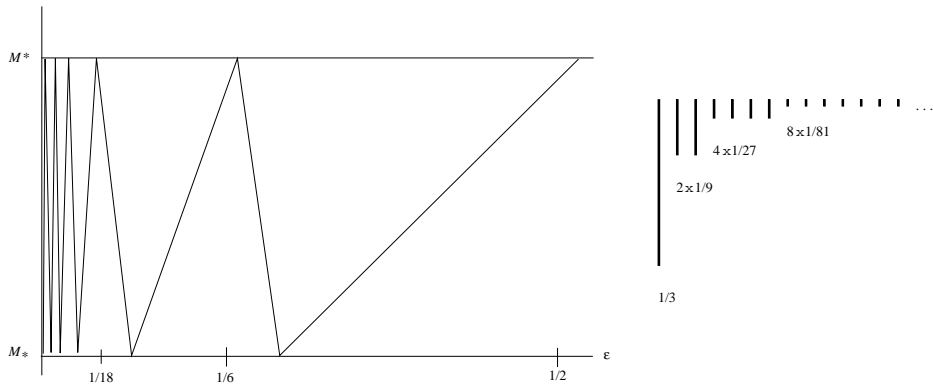
$$\zeta_\nu(s) = \zeta_{\mathcal{L}}(s)\zeta(s),$$

where  $\zeta(s)$  is the Riemann zeta function.

*Proof.*

$$\zeta_\nu(s) = \sum_{k,j=1}^{\infty} (k \cdot \ell_j^{-1})^{-s} = \sum_{j=1}^{\infty} \ell_j^s \sum_{k=1}^{\infty} k^{-s} = \zeta_{\mathcal{L}}(s)\zeta(s).$$

□

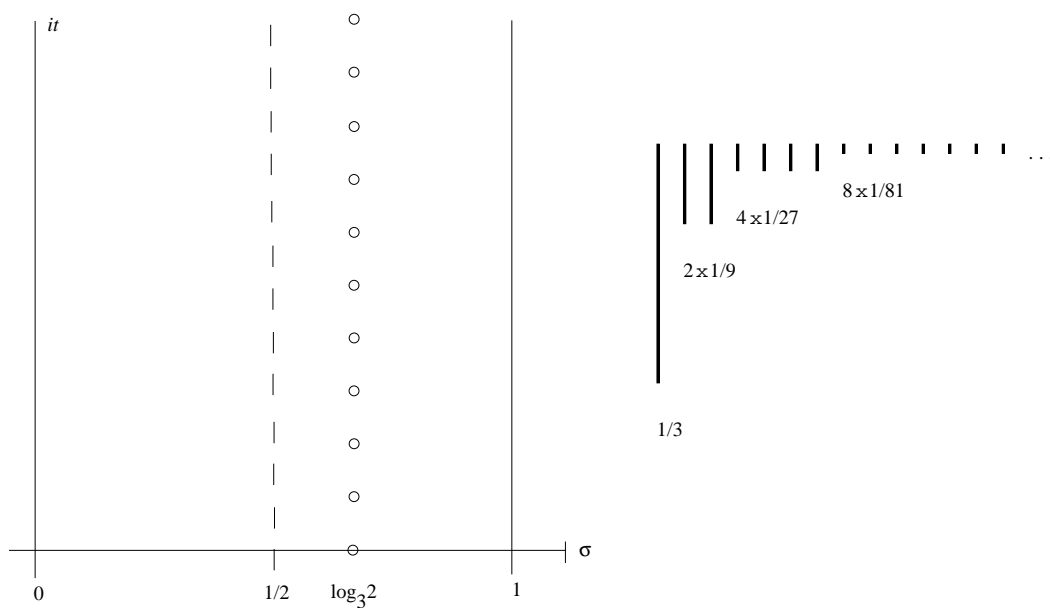


In the figure above, the value of  $V(\varepsilon)\varepsilon^{-(1-D)}$  “oscillates in dimension  $D$ ” as  $\varepsilon \rightarrow 0^+$ .

**Definition 7.** A fractal string with lengths  $\mathcal{L}$  and is Minkowski measurable if the following limit exists in  $(0, \infty)$ :

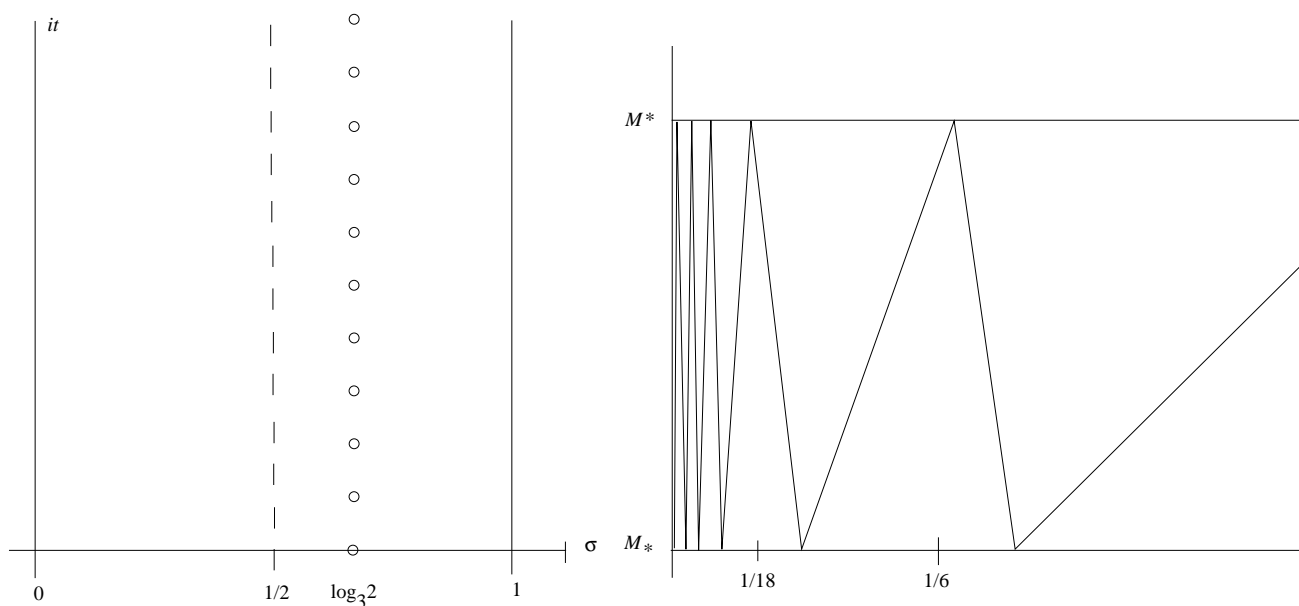
$$M = \lim_{\varepsilon \rightarrow 0^+} V(\varepsilon)\varepsilon^{-(1-D)}.$$

**The Inverse Spectral Problem (S).** Consider a given fractal string with Minkowski dimension  $D \in (0, 1)$ . If this string has no oscillations of order  $D$  in its spectrum, does it follow that it is Minkowski measurable?



**Theorem 8.** *Under certain conditions on a fractal string  $\Omega$  with lengths  $\mathcal{L}$ , the following are equivalent:*

1.  *$D$  is the only complex dimension with real part  $D$ .*
2. *The boundary of  $\Omega$  is Minkowski measurable.*



**“Can one hear if a fractal string is Minkowski measurable?”**

**Theorem 9.** *For a given  $D$  in the critical interval  $(0, 1)$ , the inverse spectral problem **(S)** in dimension  $D$  has a positive answer if and only if  $\zeta(s)$  does not have any zero on the vertical line  $\text{Re}(s) = D$ .*

**(S)** Consider a given fractal string with Minkowski dimension  $D \in (0, 1)$ . If this string has no oscillations of order  $D$  in its spectrum, does it follow that it is Minkowski measurable?

## References

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- [2] M. L. Lapidus and H. Maier, The Riemann hypothesis and inverse spectral problems for fractal strings, *J. London Math. Soc.* (2) **52** (1995), 15–34.
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- [4] J. A. Rock, Zeta Functions, Complex Dimensions of Fractal Strings and Multifractal Analysis of Mass Distributions, *Ph. D. Dissertation*, University of California, Riverside, 2007.