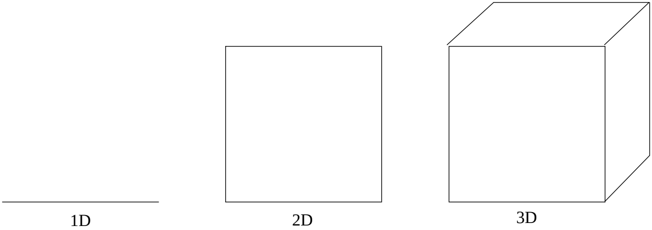
DIMENSIONS OF FRACTAL STRINGS AND MULTIFRACTAL MEASURES

John A. Rock

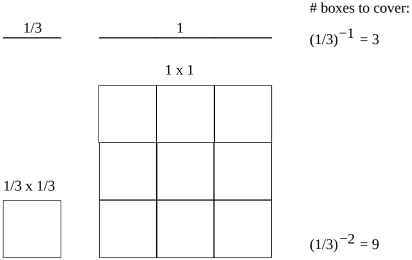
October 20th, 2009

Dimension



A typical notion of dimension: Line segments are 1D, squares are 2D, cubes are 3D…

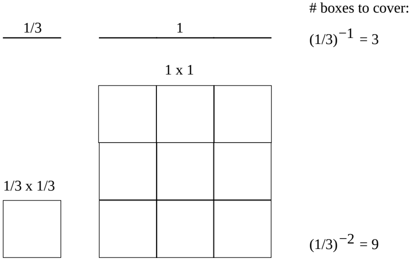
Dimension and Side Length



$$N_\varepsilon = \varepsilon^{-D} \Leftrightarrow D = \frac{\log N_\varepsilon}{-\log \varepsilon}$$

where ε is the length of the sides and N_ε is the minimum number of boxes required to a set.

Dimension and Side Length



$$N_\varepsilon = \varepsilon^{-D} \Leftrightarrow D = \frac{\log N_\varepsilon}{-\log \varepsilon}$$

where ε is the length of the sides and N_ε is the minimum number of boxes required to a set.

Box Dimension

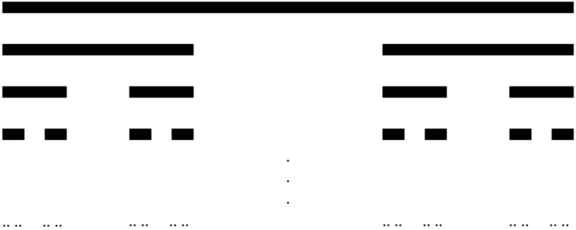
Definition

The *box dimension* $\dim_B$ *of a bounded subset* F *of* $\mathbb{R}^m$ *is given by the following limit (when it exists):*

$$\dim_B(F) = \lim_{\varepsilon \to 0^+} \frac{\log N_\varepsilon(F)}{-\log \varepsilon},$$

where $N_\varepsilon(F)$ *is the smallest number of boxes with side length* ε *that cover* F*.*

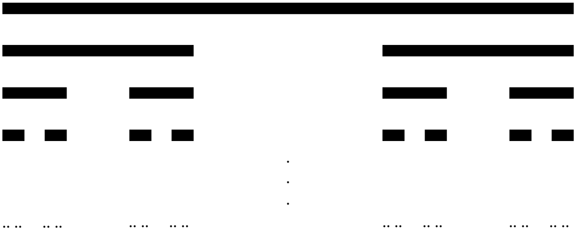
Cantor Set



The Cantor set has the following properties:

- It has length 0.
- It is a perfect subset of $[0, 1]$.
- It is uncountable.
- It is self-similar.
- It has box dimension equal to $\log_3 2$.

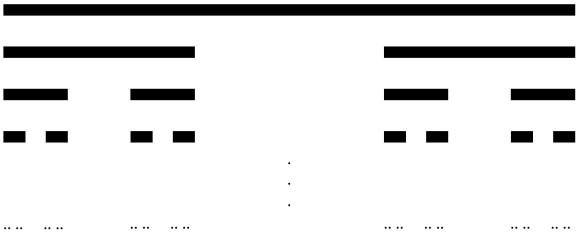
Cantor Set



The Cantor set has the following properties:

- It has length 0.
- It is a perfect subset of $[0, 1]$.
- It is uncountable.
- It is self-similar.
- It has box dimension equal to $\log_3 2$.

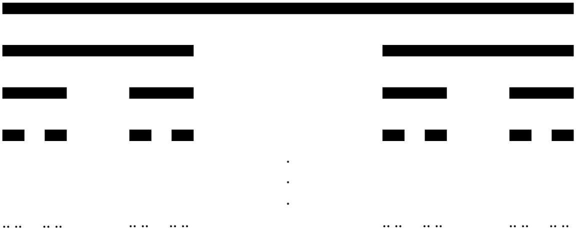
Cantor Set



The Cantor set has the following properties:

- It has length 0.
- It is a perfect subset of $[0,1]$.
- It is uncountable.
- It is self-similar.
- It has box dimension equal to $\log_3 2$.

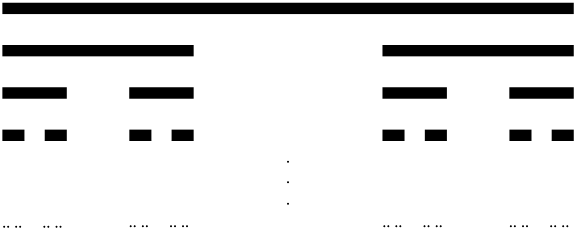
Cantor Set



The Cantor set has the following properties:

- It has length 0.
- It is a perfect subset of $[0, 1]$.
- It is uncountable.
- It is self-similar.
- It has box dimension equal to $\log_3 2$.

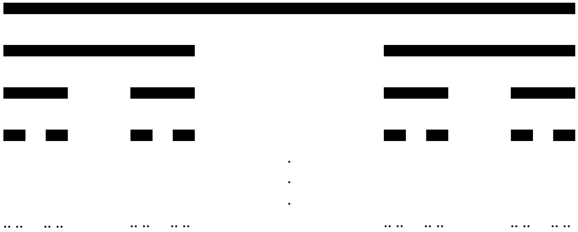
Cantor Set



The Cantor set has the following properties:

- It has length 0.
- It is a perfect subset of $[0, 1]$.
- It is uncountable.
- It is self-similar.
- It has box dimension equal to $\log_3 2$.

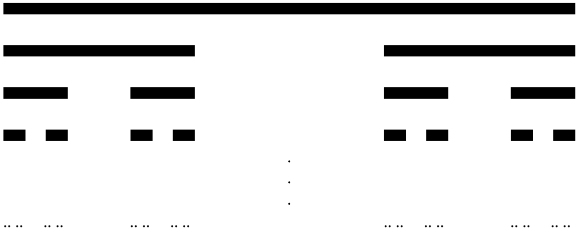
Cantor Set



The Cantor set has the following properties:

- It has length 0.
- It is a perfect subset of $[0, 1]$.
- It is uncountable.
- It is self-similar.
- It has box dimension equal to $\log_3 2$.

Non-Integer Dimension



$$\lim_{\varepsilon\to 0^+}\frac{\log N_\varepsilon(F)}{-\log\varepsilon}=\lim_{n\to\infty}\frac{\log\left(2^n\right)}{-\log\left(3^{-n}\right)}=\log_3 2\approx 0.6309$$

Fractal Strings

Definition

A fractal string Ω *is a bounded open subset of the real line. The collection of lengths* ℓ_j *of the disjoint intervals is denoted* $\mathcal{L}$.

Theorem

If a fractal string Ω *in* $[0,1]$ *is of total length 1 and has an infinite number of lengths in its sequence* $\mathcal{L}$*, then*

$$\mathrm{dim}_B(\partial\Omega) = \inf\left\{\sigma \in \mathbb{R} \;\middle|\; \sum_{j=1}^{\infty} \ell_j^\sigma < \infty\right\},$$

where $\partial\Omega = [0,1] \setminus \Omega$.

Fractal Strings

Definition

A fractal string Ω *is a bounded open subset of the real line. The collection of lengths* ℓ_j *of the disjoint intervals is denoted* $\mathcal{L}$.

Theorem

If a fractal string Ω *in* $[0,1]$ *is of total length 1 and has an infinite number of lengths in its sequence* $\mathcal{L}$*, then*

$$\dim_B(\partial\Omega) = \inf\left\{\sigma \in \mathbb{R} \mid \sum_{j=1}^{\infty} \ell_j^{\sigma} < \infty\right\},$$

where $\partial\Omega = [0,1] \setminus \Omega$.

Geometric Zeta Function

Definition

The *<u>geometric zeta function</u>* *of a fractal string* Ω *with lengths* $\mathcal{L}$ *is*

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} \ell_j^s = \sum_{n=1}^{\infty} m_n l_n^s,$$

where $\mathrm{Re}(s) > \dim_B(\partial\Omega)$*.*

For the Cantor String, the geometric zeta function is

$$\zeta_{\mathcal{L}}(s) = \zeta_{CS}(s) = \sum_{n=1}^{\infty} 2^{n-1} 3^{-ns} = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}}.$$

Geometric Zeta Function

Definition

The *<u>geometric zeta function</u>* *of a fractal string* Ω *with lengths* $\mathcal{L}$ *is*

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} \ell_j^s = \sum_{n=1}^{\infty} m_n l_n^s,$$

where $\operatorname{Re}(s) > \dim_B(\partial\Omega)$*.*

For the Cantor String, the geometric zeta function is

$$\zeta_{\mathcal{L}}(s) = \zeta_{CS}(s) = \sum_{n=1}^{\infty} 2^{n-1} 3^{-ns} = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}}.$$

Regularity

Definition

The *regularity* $A(U)$ *of a (Borel) measure* μ *on a subset* $U \subset [0,1]$ *with range in* $[0,\infty]$ *is*

$$A(U) = \frac{\log \mu(U)}{\log |U|},$$

where $|\cdot| = \lambda(\cdot)$ *is the Lebesgue measure on* $[0,1]$.

Equivalently, $A(U)$ is the exponent α that satisfies

$$|U|^{\alpha} = \mu(U).$$

Regularity

Definition

The <u>*regularity*</u> *$A(U)$ of a (Borel) measure μ on a subset $U \subset [0,1]$ with range in $[0,\infty]$ is*

$$A(U) = \frac{\log \mu(U)}{\log |U|},$$

where $|\cdot| = \lambda(\cdot)$ is the Lebesgue measure on $[0,1]$.

Equivalently, $A(U)$ is the exponent α that satisfies

$$|U|^{\alpha} = \mu(U).$$

Two Ideas

Definition

For a measure μ and ordered family of partitions $\mathfrak{P} = \{\mathcal{P}_n\}_{n=1}^{\infty}$ which mesh to zero, the partition zeta function with regularity α is

$$\zeta_{\mathfrak{P}}^{\mu}(\alpha, s) = \sum_{n=1}^{\infty} \sum_{A(P_n^k)=\alpha} |P_n^k|^s,$$

where the P_n^k are the intervals in the partition $\mathcal{P}_n$ (and $\mathrm{Re}(s)$ *is large enough).*

Definition

For a measure μ and sequence of scales $\mathcal{N}$, the *multifractal zeta function* *with regularity α is*

$$\zeta_{\mathcal{N}}^{\mu}(\alpha, s) = \sum_{n=1}^{\infty} \sum_{p=1}^{k_n(\alpha)} |K_n^p(\alpha)|^s,$$

where $k_n(\alpha)$ is the number of intervals $K_n^p(\alpha)$ (and $\mathrm{Re}(s)$ is large enough).

Binomial Measure

The first few stages in the construction of a mass distribution ν on the Cantor set. At each stage, mass is split from the previous stage in ratios of $1/3$ on the left and $2/3$ on the right.

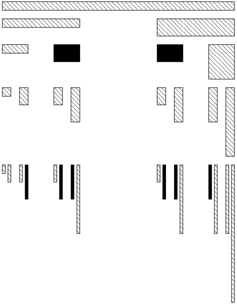
Distinct Regularity Values

For the measure ν and the family $\mathfrak{P}$ of partitions given by the open and closed intervals in the construction of the Cantor set, the regularity values are

$$\alpha = \alpha(k_1, k_2) = \frac{\log\left(2^{nk_1}/3^{nk_2}\right)}{\log\left(1/3^{nk_2}\right)} = 1 - \frac{k_1}{k_2}\log_3 2,$$

where k_1 and k_2 are relatively prime non-negative integers such that $k_1 < k_2$.

Partition Zeta Functions

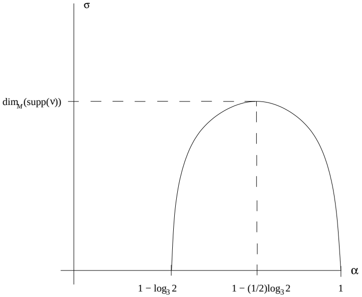


Construction of a partition zeta function:

$$\zeta_{\mathfrak{P}}^{\nu}(\alpha(k_1,k_2),s) = \sum_{n=1}^{\infty} \binom{nk_2}{nk_1} 3^{-nk_2 s},$$

for $\mathrm{Re}(s)$ large enough.

Multifractal Spectrum



The abscissa of convergence function $f(\alpha)$ is

$$\begin{aligned} f(\alpha) &= \frac{(\alpha-1)}{\log_3 2}\cdot\log_3\left(\frac{-(\alpha-1)}{\log_3 2}\right) \\ &\quad -\left(1+\frac{(\alpha-1)}{\log_3 2}\right)\cdot\log_3\left(1+\frac{(\alpha-1)}{\log_3 2}\right). \end{aligned}$$

Maximum of the Spectrum

The maximum of f is attained at $\alpha = \alpha(1,2) = 1 - (1/2)\log_3 2$ and this value coincides with the box dimension of the support of the measure ν. That is,

$$\begin{aligned} \dim_B(supp(\nu)) &= \max\{\ f(\alpha) \mid \alpha = \alpha(k_1, k_2)\} \\ &= \log_3 2. \end{aligned}$$

A Not-So-Simple Measure

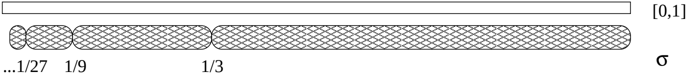


Figure: *Approximation of the measure σ.*

Let $\mathcal{N} = \{3^{-n}\}_{n=1}^{\infty}$ and

$$\sigma = \sum_{j=1}^{\infty} 3^{-j} \delta_{3^{-j}}.$$

Goal: Find all the regularity values attained by σ with intervals U that have length in $\mathcal{N}$.

A Not-So-Simple Measure

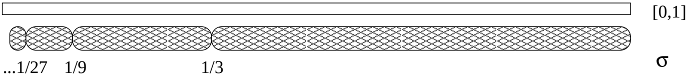


Figure: *Approximation of the measure σ.*

Let $\mathcal{N} = \{3^{-n}\}_{n=1}^{\infty}$ and

$$\sigma = \sum_{j=1}^{\infty} 3^{-j} \delta_{3^{-j}}.$$

Goal: Find all the regularity values attained by σ with intervals U that have length in $\mathcal{N}$.

Cases

The positive values of $\sigma(U)$ are obtained in one of the two following ways:

- Case 1: U contains exactly one point-mass of size 3^{-j} where $j \leq N$.
- Case 2: U contains two or more point-masses, necessarily including the point-mass 3^{-N}. If any other smaller point-mass 3^{-p} is also contained in U, so are all point-masses 3^{-j} between 3^{-p} and 3^{-N} (i.e., $N \leq j \leq p$). That is, U contains any finite or infinite sequence of point-masses $\{3^{-j}\}_{j=N}^{p}$, where $p > N$.

Cases

The positive values of $\sigma(U)$ are obtained in one of the two following ways:

- Case 1: U contains exactly one point-mass of size 3^{-j} where $j \leq N$.
- Case 2: U contains two or more point-masses, necessarily including the point-mass 3^{-N}. If any other smaller point-mass 3^{-p} is also contained in U, so are all point-masses 3^{-j} between 3^{-p} and 3^{-N} (i.e., $N \leq j \leq p$). That is, U contains any finite or infinite sequence of point-masses $\{3^{-j}\}_{j=N}^{p}$, where $p > N$.

Cases

The positive values of $\sigma(U)$ are obtained in one of the two following ways:

- Case 1: U contains exactly one point-mass of size 3^{-j} where $j \leq N$.
- Case 2: U contains two or more point-masses, necessarily including the point-mass 3^{-N}. If any other smaller point-mass 3^{-p} is also contained in U, so are all point-masses 3^{-j} between 3^{-p} and 3^{-N} (i.e., $N \leq j \leq p$). That is, U contains any finite or infinite sequence of point-masses $\{3^{-j}\}_{j=N}^{p}$, where $p > N$.

Regularity Breakdown

Proposition

For the measure σ *and sequence of scales* $\mathcal{N} = \{3^{-n}\}_{n=1}^{\infty}$, *the possible finite regularity values of* U *where* $|U| = 3^{-M}$ *for some fixed* $M \in \mathbb{N}^*$ *are:*

$$\begin{aligned}
\alpha(m_1, m_2) &= \frac{\log 3^{-m_1 n}}{\log 3^{-m_2 n}} = \frac{m_1}{m_2}, \\
\alpha_M(k) &= \frac{\log\left((3^{k-M+1} - 1)/2\right)}{-M \log 3} + \frac{k+1}{M}, \\
\alpha_M(\infty) &= 1 + \frac{\log 2}{-M \log 3},
\end{aligned}$$

where $m_1 < m_2$ *and* $(m_1, m_2) = 1$ *for* $m_1, m_2 \in \mathbb{N}^*$, *and* $k \in \mathbb{N}^*$ *such that* $k > M$. *These regularity values are all distinct from one another.*

Building a MZF

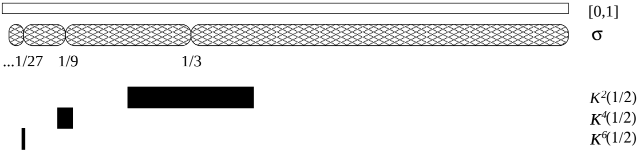


Figure: *Approximation of* σ *and the construction of* $\zeta^{\sigma}_{\mathcal{N}}(1/2, s)$.

The solid black bars represent the $K^n_p(\alpha)$ with $\alpha = 1/2$ that generate the terms of the multifractal zeta function. For $\alpha(m_1, m_2) = m_1/m_2$, only stages at multiples of m_2 have intervals with the correct regularity, hence the other stages are skipped.

Multifractal Zeta Functions

Theorem

For the measure σ and sequence $\mathcal{N} = \{3^{-n}\}_{n=1}^{\infty}$, the multifractal zeta functions have the following forms:

$$\begin{aligned} \zeta_{\mathcal{N}}^{\sigma}(-\infty, s) &= 0, \\ \zeta_{\mathcal{N}}^{\sigma}(\infty, s) &= \zeta_{\mathcal{L}_\sigma}(s) = \sum_{j=1}^{\infty} \left(\frac{2}{3^j}\right)^s = \left(\frac{2}{3}\right)^s \left(\frac{1}{1-3^{-s}}\right), \end{aligned}$$

Theorem (continued)

$$\begin{aligned}
\zeta_{\mathcal{N}}^{\sigma}(m_1/m_2, s) &= \sum_{j=1}^{\infty} \left(\frac{2}{3^{m_2 j}}\right)^{s} = \left(\frac{2}{3^{m_2}}\right)^{s} \left(\frac{1}{1-3^{-m_2 s}}\right), \\
\zeta_{\mathcal{N}}^{\sigma}(1, s) &= \left(\frac{5}{9}\right)^{s} + \sum_{j=1}^{\infty} \left(\frac{2}{3^{j+2}}\right)^{s} \\
&= \left(\frac{5}{9}\right)^{s} + \left(\frac{2}{27}\right)^{s} \left(\frac{1}{1-3^{-s}}\right),
\end{aligned}$$

where $\mathcal{L}_\sigma = \{2 \cdot 3^{-j}\}_{j=1}^{\infty}$ *is the sequence of lengths of the intervals of* Ω_σ*, and* $m_1 < m_2$ *and* $(m_1, m_2) = 1$ *for all* n, m_1*, and* $m_2 \in \mathbb{N}^*$*.*

Complex Dimensions

Definition

For a measure μ, sequence $\mathcal{N}$ which tends to zero and regularity value α, the set of complex dimensions with parameter α is given by

$$\mathcal{D}^{\mu}_{\mathcal{N}}(\alpha, W) = \{\omega \in W \mid \zeta^{\mu}_{\mathcal{N}}(\alpha, s) \text{ has a pole at } \omega\}.$$

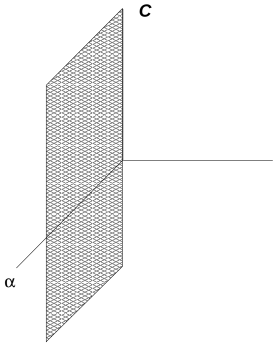
Complex Dimensions of σ

Corollary

The complex dimensions with parameter α of the measure σ and sequence $\mathcal{N} = \{3^{-n}\}_{n=1}^{\infty}$ are the poles of the multifractal zeta function $\zeta_{\mathcal{N}}^{\sigma}(\alpha, s)$. For the nontrivial values of α,

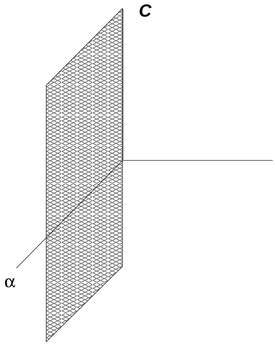
$$\begin{aligned}
\mathcal{D}_{\mathcal{N}}^{\sigma}(m_1/m_2, W) &= \left\{\omega \in W \;\middle|\; \omega \text{ is a pole of } \zeta_{\mathcal{N}}^{\sigma}\left(\frac{m_1}{m_2}, s\right)\right\} \\
&= \left\{\frac{2\pi iz}{m_2 \log 3}\right\}_{z \in \mathbb{Z}}, \\
\mathcal{D}_{\mathcal{N}}^{\sigma}(1, W) &= \{\omega \in W \mid \omega \text{ is a pole of } \zeta_{\mathcal{N}}^{\sigma}(1, s)\} \\
&= \left\{\frac{2\pi iz}{\log 3}\right\}_{z \in \mathbb{Z}},
\end{aligned}$$

for appropriate windows W.



For the measure σ and sequence $\mathcal{N}$, the full family of complex dimensions for all real α is a dense subset of a strip in $\mathbb{R} \times \mathbb{C}$. This tapestry of complex dimensions is

$$\left\{ (\alpha, \omega) | \alpha \in [0,1] \cap \mathbb{Q}, \mathrm{Re}(\omega) = 0, \mathrm{Im}(\omega) = \frac{2\pi i z}{k \log 3} \ni k \in \mathbb{N}, z \in \mathbb{Z} \right\}.$$



For the measure σ and sequence $\mathcal{N}$, the full family of complex dimensions for all real α is a dense subset of a strip in $\mathbb{R} \times \mathbb{C}$. This tapestry of complex dimensions is

$$\left\{ (\alpha, \omega) | \alpha \in [0,1] \cap \mathbb{Q}, \mathrm{Re}(\omega) = 0, \mathrm{Im}(\omega) = \frac{2\pi i z}{k \log 3} \ni k \in \mathbb{N}, z \in \mathbb{Z} \right\}.$$

References

- K. Falconer, *Fractal Geometry (Mathematical Foundations and Applications)*, 2nd ed., John Wiley, Chichester, 2003.
- M. L. Lapidus, J. Levy Vehel and J. A. Rock, Fractal strings and multifractal zeta functions, *Letters in Mathematical Physics*, No.1, 88 (2009), 101-129.
- M. L. Lapidus and J. A. Rock, Towards zeta functions and complex dimensions of multifractals, *Complex Variables and Elliptic Equations*, special issue dedicated to fractals, No.6, 54 (2009), 545-559.
- M. L. Lapidus and M. van Frankenhuijsen, *Fractal Geometry, Complex Dimensions and Zeta Functions*, Springer 2006.
- J. A. Rock, *Zeta Functions, Complex Dimensions of Fractal Strings, and Multifractal Analysis of Mass Distributions*, thesis, June 2007.