Fractal Strings and Multifractal Zeta Functions

Michel L. Lapidus, Jacques Lévy-Véhel and John A. Rock

August 4, 2006

Abstract. We define a one-parameter family of geometric zeta functions for a Borel measure on the unit interval and a sequence which tends to zero. The construction of this family is based on that of the continuous large deviation spectra in multifractal analysis. For a measure which is singular with respect to the Lebesgue measure and a naturally chosen sequence, a certain value of the parameter yields the fractal string and geometric zeta function of the complement of the support of the measure. This new family of zeta functions yields topological and multifractal information which is absent in the current theory of fractal strings.

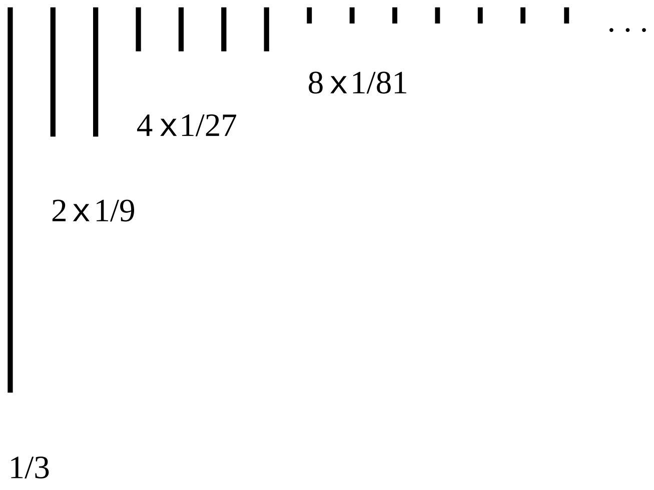

Figure 1: *The lengths of the Cantor String.*

1 Fractal Strings

Definition 1.1. *A fractal string* Ω *is a bounded open subset of the real line.*

The volume of the inner-ε neighborhood of the boundary of Ω is

$$V(\varepsilon) = \text{vol}_1\{x \in \Omega : d(x, \partial\Omega) < \varepsilon\}.$$

The *Minkowski dimension* of a fractal string Ω with non-increasing sequence of lengths $\mathcal{L} = \{\ell_j\}_{j=1}^{\infty}$ is

$$D_{\mathcal{L}} := \inf\{\alpha \geq 0 : V(\varepsilon) = O(\varepsilon^{1-\alpha}) \text{ as } \varepsilon \to 0^+\}.$$

We have

$$D_{\mathcal{L}} = \inf\{\sigma \in \mathbb{R} : \sum_{j=1}^{\infty} \ell_j^{\sigma} < \infty\}.$$

Definition 1.2. *The geometric zeta function of a fractal string* Ω *with lengths* $\mathcal{L}$ *is*

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} \ell_j^s = \sum_{n=1}^{\infty} m_n l_n^s$$

where $\mathrm{Re}(s) > D_{\mathcal{L}}$.

Definition 1.3. *The set of complex dimensions of a fractal string* Ω *with lengths* $\mathcal{L}$ *is*

$$\mathcal{D}_{\mathcal{L}}(W) = \{\omega \in W : \zeta_{\mathcal{L}} \text{ has a pole at } \omega\}.$$

Theorem 1.1. *The volume of the one-sided tubular neighborhood of radius* ε *of the boundary of* Ω *(with lengths* $\mathcal{L}$*) is given by the following distributional explicit formula (with error term) on the space of test functions* $\mathbf{D}(0,\infty)$:

$$V(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathcal{L}}(W)} \mathrm{res}\left(\frac{\zeta_{\mathcal{L}}(s)(2\varepsilon)^{1-s}}{s(1-s)}; \omega\right) + \mathcal{R}(\varepsilon)$$

2 Multifractal Analysis

Let $\mathbf{X}([0,1])$ denote the space of closed subintervals of $[0,1]$.

Definition 2.1. *The* <u>*regularity*</u> *of a Borel measure* μ *on* $U \in \mathbf{X}([0,1])$ *is*

$$A(U) = \frac{\log \mu(U)}{\log |U|},$$

where $|\cdot| = m_L(\cdot)$ *is the Lebesgue measure.*

Definition 2.2. *The* <u>*large deviation spectrum*</u> *is*

$$f_g(\alpha) = \lim_{\varepsilon \to 0} \left(\limsup_{n \to \infty} \frac{\log N_\alpha(\varepsilon, n)}{n \log 2} \right)$$

with the convention that $\log N_\alpha(\varepsilon, n)/n \log 2 = -\infty$ *if* $N_\alpha(\varepsilon, n) = 0$.

$$\mathcal{R}_\eta(\alpha) = \{U \in \mathbf{X}([0,1]) \vdots |U| = \eta \text{ and } A(U) = \alpha\}.$$

Definition 2.3. *The* *continuous large deviation spectrum* *is*

$$\begin{aligned}\tilde{f}_g^c(\alpha) &= \limsup_{\eta\to 0} \frac{\log\left(|\bigcup_{U\in\mathcal{R}_\eta(\alpha)} U|\right)}{|\log \eta|} \\ &= \limsup_{\eta\to 0}\left(1 - \frac{\log|\bigcup_{U\in\mathcal{R}_\eta(\alpha)} U|}{|\log\eta|}\right).\end{aligned}$$

Proposition 2.1. *If* μ *is a multinomial measure, then* $\tilde{f}_g^c = f_g$.

3 Definition of Multifractal Zeta Function

$$R^{\eta_n}(\alpha) = R^n(\alpha) = \bigcup_{U \in \mathcal{R}_{\eta_n}(\alpha)} U.$$

We have

$$R^n(\alpha) = \bigcup_{i=1}^{r_n(\alpha)} R_i^n(\alpha),$$

where $r_n(\alpha)$ is the number of connected components $R_i^n(\alpha)$ of $R^n(\alpha)$. We denote the endpoints of the $R_i^n(\alpha)$ by $R_i^n(\alpha) = (a_R^n(\alpha, i), b_R^n(\alpha, i))$.

$$J^1(\alpha) = R^1(\alpha),$$

$$J^n(\alpha) = R^{n-1}(\alpha) \ominus R^n(\alpha), n \geq 2.$$

$$K^n(\alpha) = \bigcup_{i=1}^{k_n(\alpha)} K_i^n(\alpha) \subset J^n(\alpha),$$

The $K_i^n(\alpha)$ are the $J_i^n(\alpha)$ such that $a_J^n(\alpha, i) \neq a_R^n(\alpha, j)$ and $b_J^n(\alpha, i) \neq b_R^n(\alpha, j)$ for all $i \in \{1, ..., j_n(\alpha)\}$ and $j \in \{1, ..., r_n(\alpha)\}$.

Let

$$\mathcal{K}_{\mathcal{N}}^{\mu}(\alpha) = \{|K_i^n(\alpha)| \vdots n \in \mathbb{N},\ i \in \{1, ..., k_n(\alpha)\}\}.$$

Definition 3.1. *The multifractal zeta function of a measure μ and a sequence $\mathcal{N}$ is*

$$\zeta_{\mathcal{N}}^{\mu}(\alpha, s) = \sum_{n=1}^{\infty} \sum_{i=1}^{k_n(\alpha)} |K_i^n(\alpha)|^s = \zeta_{\mathcal{K}_{\mathcal{N}}^{\mu}(\alpha)}(s).$$

Definition 3.2. *For a measure μ, sequence $\mathcal{N}$ which tends to zero and regularity value α,*

$$\mathcal{D}_{\mathcal{N}}^{\mu}(\alpha, W) = \{\omega \in W \vdots \zeta_{\mathcal{N}}^{\mu}(\alpha, s) \text{ has a pole at } \omega\}.$$

4 A Result for Singular Measures

Theorem 4.1. *For* $\sigma \perp m_L$, $\mu_\sigma = m_L + |\sigma|$ *and* $\mathcal{N}$ *such that* $l_n > \eta_n > l_{n+1}$,

$$\zeta_{\mathcal{N}}^{\mu_\sigma}(1, s) = \zeta_{\mathcal{L}_\sigma}(s).$$

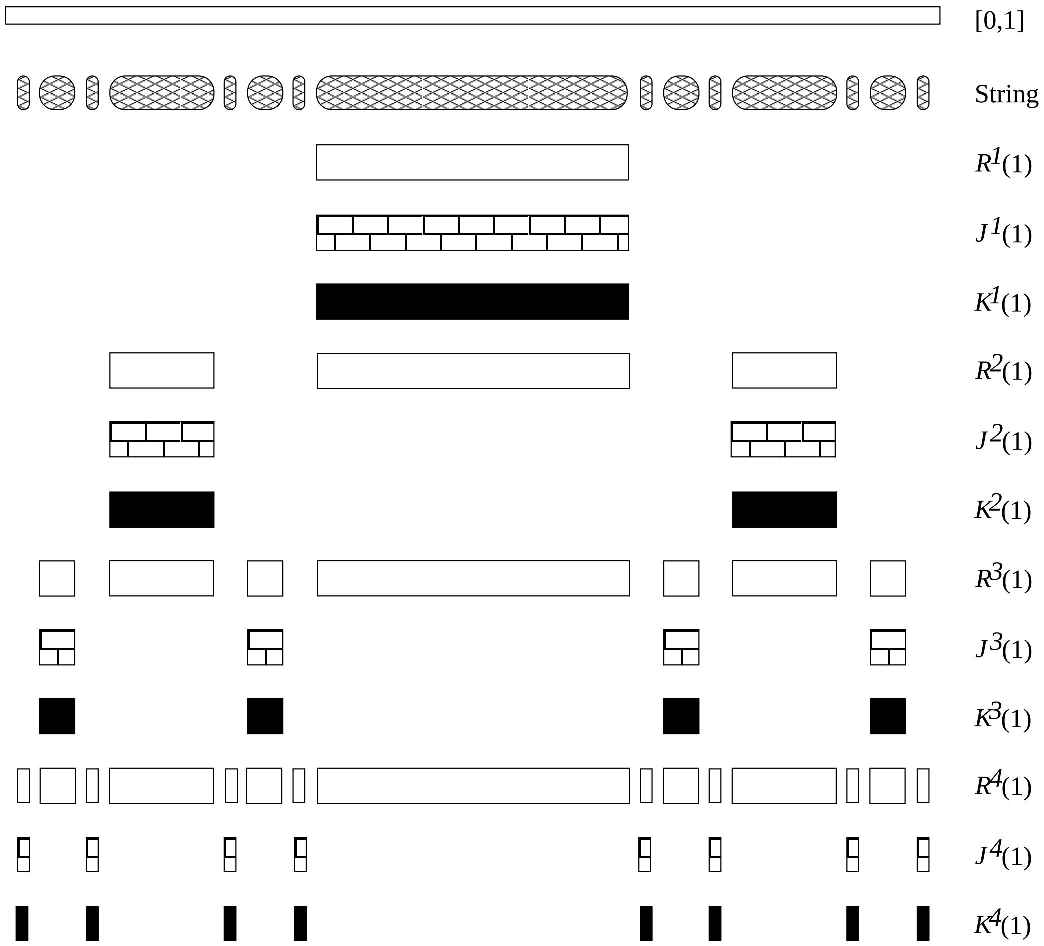

Figure 2: *This is the first four stages of the construction of the multifractal zeta function* $\zeta_{\mathcal{N}}^{\mu_1}(1,s)$ *in the proof of Theorem 4.1 applied to a measure* μ_1 *and sequence of scales* $\mathcal{N} = \{3^{-n-1}\}_{n=1}^{\infty}$*. The solid black bars represent the lengths used to construct the multifractal zeta function* $\zeta_{\mathcal{N}}^{\mu_1}(1,s)$*.*

5 Fractal Strings and Unit Point-Mass

Theorem 5.1. *For a fractal string* Ω *with sequence of lengths* $\mathcal{L}$ *and perfect boundary, consider*

$$\mu_\Omega = m_L + \sum_{j=1}^{\infty}(\delta_{a_j} + \delta_{b_j}).$$

Suppose there exists a sequence $\mathcal{N}$ *such that* $l_n > \eta_n \geq l_{n+1}$ *and* $l_n > 2\eta_n$*. Then,*

$$\begin{aligned} \zeta_{\mathcal{N}}^{\mu_\Omega}(1,s) &= \zeta_{\mathcal{L}}(s) \quad \textit{and} \\ \zeta_{\mathcal{N}}^{\mu_\Omega}(-\infty,s) &= h(s) + \sum_{n=2}^{\infty} m_n(l_n - 2\eta_n)^s, \end{aligned}$$

where $h(s)$ *is the entire function given by* $h(s) = \sum_{i=1}^{k_1(-\infty)} |K_i^1(-\infty)|^s$.

Corollary 5.1.1. *For a fractal string Ω with perfect boundary, total length 1, lengths $\mathcal{L}$ given by $l_n = ca^{-n}$ with multiplicities m_n such that $a > 2$ and c is a normalization constant, and given a sequence of scales $\mathcal{N}$ where $\eta_n = l_{n+1} = ca^{-n-1}$,*

$$\zeta_{\mathcal{N}}^{\mu_\Omega}(-\infty, s) = f_0(s) + f_1(s)\zeta_{\mathcal{L}}(s),$$

where $f_0(s)$ and $f_1(s)$ are entire.

Proof: By Theorem 5.1,

$$\begin{aligned}
\zeta_{\mathcal{N}}^{\mu_\Omega}(-\infty, s) &= h(s) + \sum_{n=2}^{\infty} m_n (l_n - 2l_{n+1})^s \\
&= h(s) + c^s \sum_{n=2}^{\infty} m_n (a^{-n} - 2a^{-n-1})^s \\
&= h(s) + c^s \sum_{n=2}^{\infty} m_n \left(\frac{a-2}{a^{n+1}} \right)^s \\
&= h(s) + c^s \left(\frac{a-2}{a} \right)^s \sum_{n=2}^{\infty} m_n a^{-ns} \\
&= h(s) + c^s \left(\frac{a-2}{a} \right)^s \left(\zeta_{\mathcal{L}}(s) - m_1 a^{-s} \right).
\end{aligned}$$

□

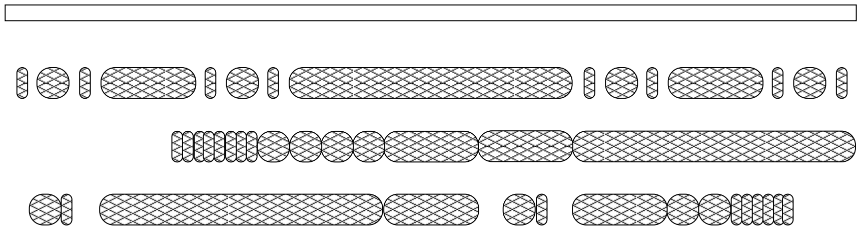

Figure 3: *Three fractal strings with the same lengths as the Cantor String.*

6 Three Cantor Strings

$$\mathcal{D}_{CS}(\mathbb{C}) = \{\log_3 2 + \frac{2i\pi k}{\log 3} : k \in \mathbb{Z}\}.$$

These are the poles of

$$\zeta_{\mathcal{N}}^{\mu_i}(1,s) = \zeta_{CS}(s) = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}},$$

where

$$\mu_i = \mu_{\Omega_i}$$

are the measures with unit point-masses at the endpoint of each interval comprising the strings Ω_i.

Consider the multifractal zeta functions corresponding these measures at regularity $-\infty$. In each case we use $\mathcal{N} = \{3^{-n-1}\}_{n=1}^{\infty}$.

$$\begin{aligned} dim_H(\partial\Omega_1) &= D_{\mathcal{CS}} = \log_3 2, \\ dim_H(\partial\Omega_2) &= 0, \\ dim_H(\partial\Omega_3) &= \log_9 2. \end{aligned}$$

$$\zeta_{\mathcal{N}}^{\mu_1}(-\infty, s) = 2\left(\frac{4}{9}\right)^s + \frac{2}{27^s}\left(\frac{1}{1 - 2\cdot 3^{-s}}\right).$$

The set of poles of $\zeta_{\mathcal{N}}^{\mu_1}(-\infty, s)$ is

$$\mathcal{D}_{\mathcal{N}}^{\mu_1}(-\infty) = \left\{\log_3 2 + \frac{2i\pi k}{\log 3}\right\}_{k\in\mathbb{Z}} = \mathcal{D}_{CS}(\mathbb{C}).$$

$$\zeta_{\mathcal{N}}^{\mu_2}(-\infty, s) = \frac{1}{9^s},$$

which, of course, is entire and has no poles.

$$\zeta_{\mathcal{N}}^{\mu_3}(-\infty, s) = h_3(s) + \left(\frac{2^{s+1}}{81^s}\right)\left(\frac{1}{1 - 2\cdot 9^{-s}}\right)$$

where $h_3(s)$ is entire. The set of poles of $\zeta_{\mathcal{N}}^{\mu_5}(-\infty, s)$ is

$$\mathcal{D}_{\mathcal{N}}^{\mu_3}(-\infty) = \left\{\log_9 2 + \frac{2i\pi k}{\log 9}\right\}_{k\in\mathbb{Z}}.$$

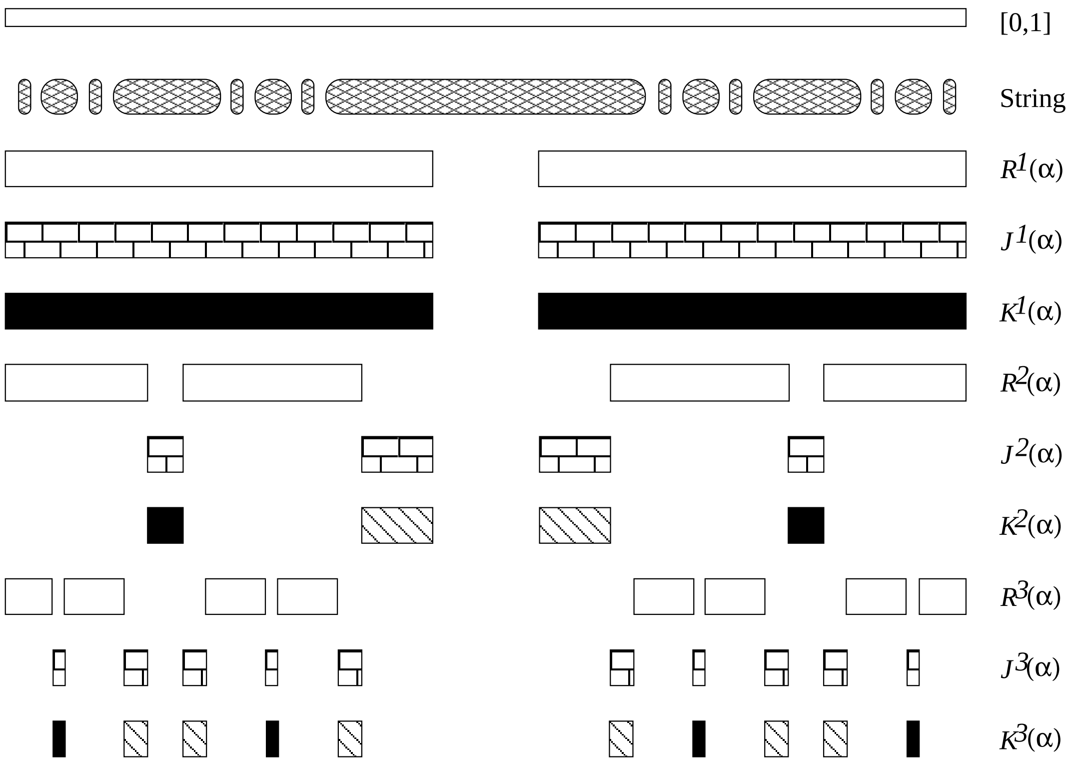

Figure 4: *The first three stages in the construction of* $\zeta_{\mathcal{N}}^{\mu_1}(-\infty, s)$ *where* $\mathcal{N}$ *is the lengths of the Cantor String beginning with* $1/9$. *The measure* μ_1 *is supported on the Cantor Set. The solid black bars represent the lengths used to construct the multifractal zeta function* $\zeta_{\mathcal{N}}^{\mu_1}(-\infty, s)$.

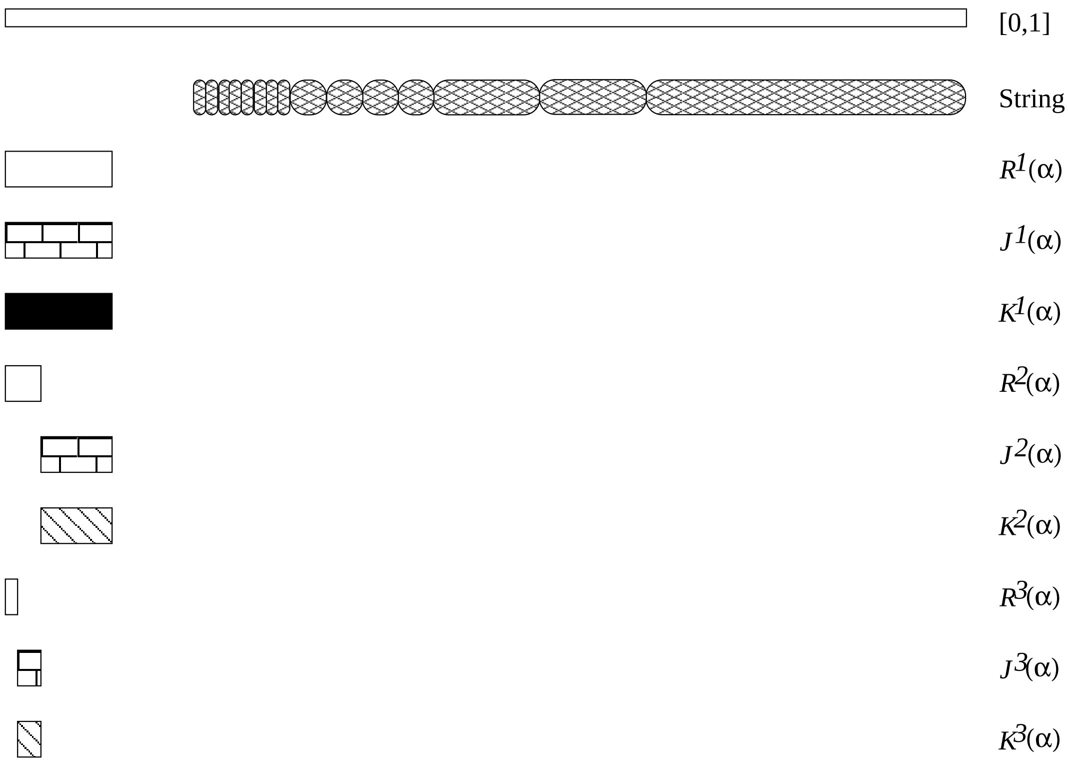

Figure 5: *The first three stages in the construction of* $\zeta_{\mathcal{N}}^{\mu_2}(-\infty, s)$ *where* $\mathcal{N}$ *is the lengths of the Cantor String beginning with* $1/9$. *The measure* μ_2 *is supported on a set with a single accumulation point at* 0. *The solid black bars represent the lengths used to construct the multifractal zeta function* $\zeta_{\mathcal{N}}^{\mu_2}(-\infty, s)$. *Note that only the first stage contributes to the construction.*

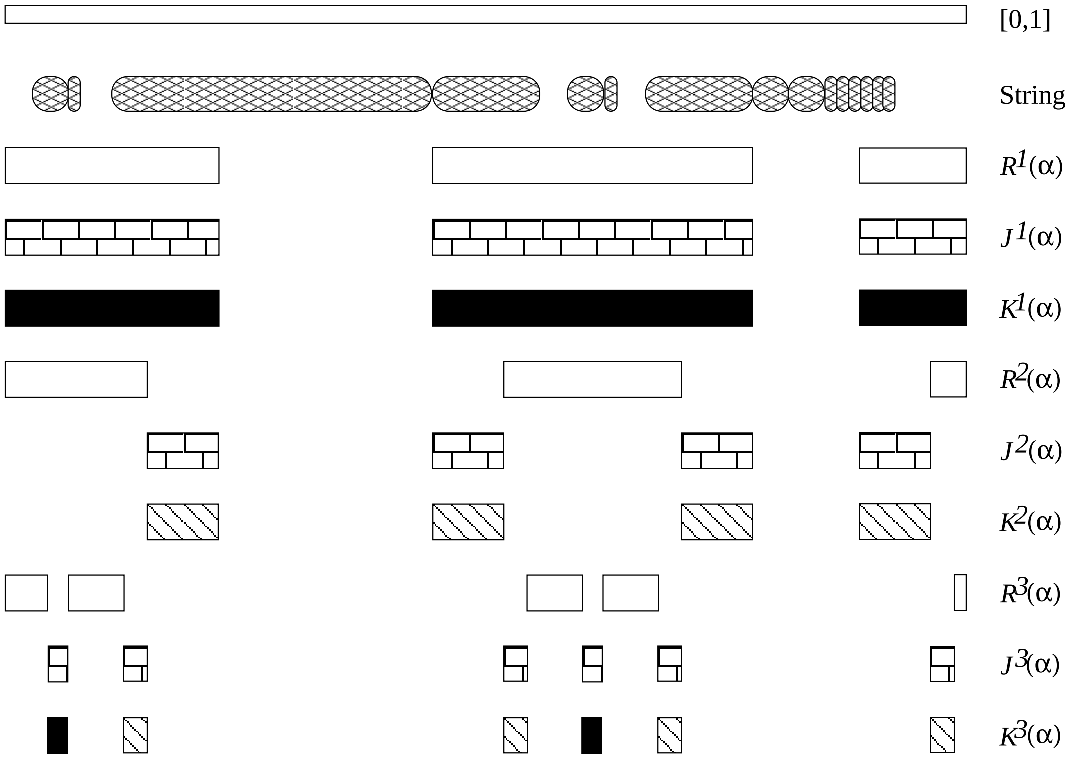

Figure 6: *The first three stages in the construction of* $\zeta_{\mathcal{N}}^{\mu_3}(-\infty, s)$ *where* $\mathcal{N}$ *is the lengths of the Cantor String beginning with* $1/9$. *The measure* μ_3 *is supported on a Cantor-like set united with a set that has single accumulation point at* 1 *and countably many isolated points. The solid black bars represent the lengths used to construct the multifractal zeta function* $\zeta_{\mathcal{N}}^{\mu_3}(-\infty, s)$.

References

[1] M. Arbeiter and N. Patzschke. Random self-similar multifractals. *Math. Nachr.*, 181:5–42, 1996.

[2] A. S. Besicovitch and S. J. Taylor, On the complementary intervals of a linear closed set of zero Lebesgue measure, *J. London Math. Soc.*, **29** (1954), 449-459.

[3] G. Brown, G. Michon, and J. Peyrière. On the multifractal analysis of measures. *J. Statist. Phys.*, 66(3-4):775–790, 1992.

[4] R.S. Ellis, Large Deviations for a General Class of Random Vectors, Ann. Prob., 12(1):1–12, 1984.

[5] K. Falconer, Fractal Geometry - Mathematical foundations and applications, John Wiley, Second Edition, 2003,.

[6] U. Frisch, G. Parisi

[7] B.M. Hambly and M. L. Lapidus, Random fractal strings: their zeta functions, complex dimensions and spectral asymptotics.

[8] C. Q. He and M. L. Lapidus, Generalized Minkowski content, spectrum of fractal drums, fractal strings and the Riemann zeta-function, Memoirs Amer. Math. Soc., No. 608, 127 (1997), 197.

[9] M. L. Lapidus, Fractal drum, inverse spectral problems for elliptic operators and a partial resolution of the WeylBerry conjecture, Trans. Amer. Math. Soc. 325 (1991), 465529.

[10] M. L. Lapidus, Spectral and fractal geometry: From the WeylBerry conjecture for the vibrations of fractal drums to the Riemann zeta-function, in: Differential Equations and Mathematical Physics (C. Bennewitz, ed.), Proc. Fourth UAB Internat. Conf. (Birmingham, March 1990), Academic Press, New York, 1992, pp. 151182.

[11] M. L. Lapidus, Vibrations of fractal drums, the Riemann hypothesis, waves in fractal media, and the WeylBerry conjecture, in: Ordinary and Partial Differential Equations (B. D. Sleeman and R. J. Jarvis, eds.), vol. IV, Proc. Twelfth Internat. Conf. (Dundee, Scotland, UK, June 1992),

Pitman Research Notes in Math. Series, vol. 289, Longman Scientific and Technical, London, 1993, pp. 126209.

[12] M. L. Lapidus, Fractals and vibrations: Can you hear the shape of a fractal drum?, Fractals 3, No. 4 (1995), 725736. (Special issue in honor of Benot B. Mandelbrots 70th birthday.)

[13] M. L. Lapidus and H. Maier, The Riemann hypothesis and inverse spectral problems for fractal strings, J. London Math. Soc. (2) 52 (1995), 1534.

[14] M. L. Lapidus and C. Pomerance, The Riemann zeta-function and the one-dimensional WeylBerry conjecture for fractal drums, Proc. London Math. Soc. (3) 66 (1993), 4169.

[15] M. L. Lapidus and C. Pomerance, Counterexamples to the modified WeylBerry conjecture on fractal drums, Math. Proc. Cambridge Philos. Soc. 119 (1996), 167178.

[16] M. L. Lapidus and M. van Frankenhuijsen, Fractal Geometry and Number Theory: Complex dimensions of fractal strings and zeros of zeta functions, Birkhauser, Boston, 2000.

[17] M. L. Lapidus and M. van Frankenhuijsen, A prime orbit theorem for self-similar flows and Diophantine approximation, Contemporary Mathematics 290 (2001), 113138.

[18] M. L. Lapidus and M. van Frankenhuijsen, Complex dimensions of self-similar fractal strings and Diophantine approximation, J. Experimental Mathematics, No. 1, 42 (2003), 4369.

[19] M. L. Lapidus and M. van Frankenhuijsen, Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and spectra of fractal strings, Springer Monographs in Mathematics, Springer-Verlag, New York, 2006.

[20] K.S. Lau and S.M. Ngai. L^q spectrum of the Bernouilli convolution associated with the golden ration. *Studia Math.*, 131(3):225–251, 1998.

[21] J. Lévy Véhel. Introduction to the multifractal analysis of images. *Fractal Images Encoding and Analysis, Y. Fisher, Ed.*, Springer Verlag, 1998.

[22] J. Lévy Véhel and R. Riedi. Fractional Brownian motion and data traffic modeling: The other end of the spectrum. *Fractals in Engineering, Lévy Véhel, J. Lutton, E. and Tricot, C., Eds.*, Springer Verlag, 1997.

[23] J. Lévy Véhel and C. Tricot, *On Various Multifractal Sprectra.*.

[24] J. Lévy Véhel and R. Vojak. Multifractal analysis of Choquet capacities. *Adv. in Appl. Math.*, 20(1):1–43, 1998.

[25] B.B. Mandelbrot, *Intermittent turbulence in self-similar cascades: divergence of hight moments and dimension of the carrier.* J. fluid. Mech., **62** (1974), 331–358.