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Zeta Functions, Complex Dimensions of Fractal Strings and Multifractal Analysis of Mass Distributions

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This dissertation is dedicated to my wife, Yu-Tzu,
for enriching my life with her laughter, strength and love.

ABSTRACT OF THE DISSERTATION

Zeta Functions, Complex Dimensions of Fractal Strings and Multifractal Analysis of Mass Distributions

by

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Dr. Michel L. Lapidus, Chairperson

Geometric zeta functions and complex dimensions provide a wealth of information regarding the geometry and frequency spectra of fractal strings. Multifractal zeta functions generalize these notions in the context of measures in a manner which illuminates topological structure not detected by current theory. Multifractal zeta functions also provide a new approach to the multifractal analysis of mass distributions. Partition zeta functions provide an approach to multifractal analysis which is similar to that of multifractal zeta functions, but is easier to manipulate and more faithfully reflects some of the results in current theory.

0 Introduction

This text revisits the results of [20] which introduces a generalization of geometric zeta functions called *multifractal zeta functions.* These functions are defined in the context of measures and detect topological properties of fractal strings which extend existing results and theory. This text also begins an investigation of the original purpose of the multifractal zeta functions: To provide a new approach of multifractal analysis which extends or relates to current approaches and results. The investigation leads to yet another family of functions called *partition zeta functions* which reflect much of structure apparent in current approaches to multifractal analysis.

Section 1 provides a brief review of fractal strings and geometric zeta functions, along with a description of a few examples which will be used throughout the paper, including the Cantor String and the a-String. Work on fractal strings can be found in [2, 10, 11, 17, 18, 19, 21, 24, 25] and work on geometric zeta functions and complex dimensions can be found in [26, 27, 28, 29].

Section **??** provides a brief review of some approaches to multifractal analysis that can be found in [35]. One lends itself to the definition of the multifractal zeta functions. Other approaches to multifractal analysis can be found in [1, 3, 4, 6, 7, 8, 12, 13, 14, 15, 16, 30, 31, 32, 33, 34, 36, 37, 38, 39, 40, 41, 42].

Section 2 contains the definition of one of the main objects of study, the multifractal zeta function.

Section 3 provides the statement and proof of a theorem describing the recovery of the geometric zeta function of a fractal string for parameter value $\alpha = \infty$, a result which is completely analogous to a main result of [20].

Section 4 describes the use multifractal zeta functions to investigate the properties of a given fractal string by considering a related object called the *effective string*, which in turn is related to the *effective lengths* from [20].

Section 6 investigates the properties of a certain measure ν which was the motivation for many of the measures in this text and in [20].

Section 7 investigates the family of multifractal zeta functions for a certain measure γ. This family contains a countably infinite collection of functions with poles. Further, the support of the measure γ is an interesting example of a fractal string whose structure differs significantly from the others in the text.

Section 8 analyzes the structure of multifractal measures which are supported on fractal sets or on the entire unit interval and which have fractal structure in other ways. The difficulties in the construction of the families of multifractal zeta functions of such measures become apparent. Lastly, a new type of zeta function, the *partition zeta function*, is defined to help deal with these difficulties and used to investigate these multifractal measures.

Section 9 concludes the paper with a summary of results obtained herein and a few words on possible topics for future research.

1 Fractal Strings and Geometric Zeta Functions

The first topics to review are some of the current results on fractal strings, geometric zeta functions and complex dimensions (all of which are defined below). Results on fractal strings can be found in [2, 10, 11, 17, 18, 19, 20, 21, 24, 25] and results on geometric zeta functions and complex dimensions can be found in [20, 26, 27, 28, 29].

Definition 1.1. *A fractal string Ω is a bounded open subset of the real line.*

As in [20], it will be necessary to distinguish between a fractal string Ω and its sequence of lengths $\mathcal{L}$ (with multiplicities). The sequence $\mathcal{L} = \{\ell_j\}_{j=1}^{\infty}$ is the nonincreasing sequence of lengths of the disjoint open intervals (a_j, b_j) where $\Omega = \cup_{j=1}^{\infty}(a_j, b_j)$. It will be necessary to consider the sequence of *distinct* lengths, denoted $\{l_n\}_{n=1}^{\infty}$, and their multiplicities $\{m_n\}_{n=1}^{\infty}$.

Two examples of fractal strings used extensively in this work are the a-String and the Cantor String, both of which can be found in [26, 29]. The lengths of the Cantor String (the complement of the usual ternary Cantor Set in the unit interval $[0, 1]$) appear in Figure 1.

A generalization of Minkowski dimension called *complex dimensions* are used to study the properties of certain fractal subsets of $\mathbb{R}$. For instance, the boundary of a fractal string Ω, denoted $\partial\Omega$, can be studied using complex dimensions. Throughout this text, the fractal string Ω is taken to be an open subset of the unit interval $[0, 1]$ with $\mathcal{L}$ as its associated sequence of lengths.

The one-sided volume of the tubular neighborhood of radius ε of $\partial\Omega$ is

$$V(\varepsilon) = \lambda(\{x \in \Omega \mid dist(x, \partial\Omega) < \varepsilon\}),$$

where $\lambda(\cdot) = |\cdot|$ denotes the Lebesgue measure. The *Minkowski dimension* of $\partial\Omega$, or

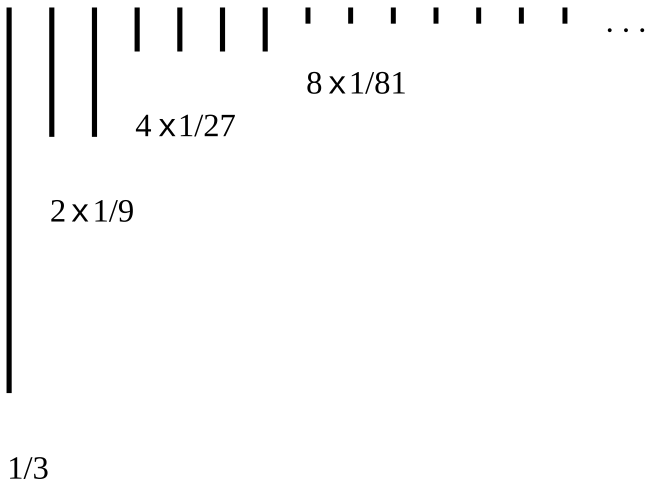

Figure 1: *The lengths of the Cantor String.*

simply of $\mathcal{L}$, is

$$\dim_M(\partial\Omega) = D = D_{\mathcal{L}} := \inf\{\alpha \geq 0 \mid \limsup_{\varepsilon\to 0^+} V(\varepsilon)\varepsilon^{\alpha-1} < \infty\}.$$

Note that one may refer directly to the Minkowski dimension of the sequence of lengths $\mathcal{L}$. This is because Minkowski dimension depends only the lengths $\mathcal{L}$.

If $\lim_{\varepsilon\to 0^+} V(\varepsilon)\varepsilon^{\alpha-1}$ exists, is positive and finite for some α, then $\alpha = D$ and we say that $\mathcal{L}$ is *Minkowski measurable.* The *Minkowski content* of $\mathcal{L}$ is then defined by $\mathcal{M}(D,\mathcal{L}) := \lim_{\varepsilon\to 0^+} V(\varepsilon)\varepsilon^{D-1}$. The Minkowski dimension is also known as the *box-counting dimension* because, for a bounded subset F of $\mathbb{R}^d$, it can be computed as

$$\limsup_{\varepsilon\to 0^+} \frac{N_\varepsilon(F)}{-\log\varepsilon},$$

where $N_\varepsilon(F)$ is the smallest number of cubes with side length ε that cover F. In [17], it is shown that if $F = \partial\Omega$ is the boundary of a bounded open set Ω, then $d-1 \leq \dim_H(F) \leq \dim_M(F) \leq d$ where d is the Euclidean dimension of the ambient space, $\dim_H(F)$ is the Hausdorff dimension of F and $\dim_M(F) = D$ is the Minkowski dimension of F. In this paper $d = 1$ and $0 \leq \dim_H(F) \leq \dim_M(F) \leq 1$.

The following equality describes a relationship between the Minkowski dimension of a fractal string Ω (really the Minkowski dimension of $\partial\Omega$) and the sum of each of its lengths with exponent $\sigma \in \mathbb{R}$. This was first observed in [18] using a key result of Besicovitch and Taylor [2], and a direct proof can be found in [29], pp. 17–18.

$$\dim_M(\partial\Omega) = D = D_{\mathcal{L}} = \inf\left\{\sigma \in \mathbb{R} \mid \sum_{j=1}^{\infty} \ell_j^\sigma < \infty\right\}.$$

$D_{\mathcal{L}}$ can be considered to be the abscissa of convergence of the Dirichlet series $\sum_{j=1}^{\infty} \ell_j^s$, where $s \in \mathbb{C}$. This Dirichlet series is the *geometric zeta function* of $\mathcal{L}$ and it is the function that has been generalized in [20] using notions from multifractal analysis.

Definition 1.2. *The geometric zeta function of a fractal string* Ω *with lengths* $\mathcal{L}$ *is*

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} \ell_j^s = \sum_{n=1}^{\infty} m_n l_n^s,$$

where $\mathrm{Re} s > D_{\mathcal{L}}$.

To consider lengths $\ell_j = 0$, the convention $0^s = 0$ for all $s \in \mathbb{C}$ is used.

One can extend the notion of the dimension of a fractal string Ω to complex values by considering the poles of $\zeta_{\mathcal{L}}$. In general, $\zeta_{\mathcal{L}}$ may not have an analytic continuation to all of $\mathbb{C}$, so consider regions where $\zeta_{\mathcal{L}}$ has a meromorphic extension and collect the poles in these regions. Specifically, consider the *screen* S where

$$S = r(t) + it,$$

for some continuous function $r : \mathbb{R} \to [-\infty, D_{\mathcal{L}}]$ and consider the *window* W which are the complex numbers to the right of the screen. That is,

$$W = \{s \in \mathbb{C} \mid \mathrm{Re} s \geq r(\mathrm{Im} s)\}.$$

Assume that $\zeta_{\mathcal{L}}$ has a meromorphic extension to an open neighborhood of W and there is no pole of $\zeta_{\mathcal{L}}$ on S.

Definition 1.3. *The set of complex dimensions of a fractal string* Ω *with lengths* $\mathcal{L}$ *is*

$$\mathcal{D}_{\mathcal{L}}(W) = \{\omega \in W \mid \zeta_{\mathcal{L}} \text{ has a pole at } \omega\}.$$

The following theorem characterizes Minkowski measurability and can be found in [26, 29].

Theorem 1.4. *If a fractal string* Ω *with lengths* $\mathcal{L}$ *satisfies certain mild conditions, the following are equivalent:*

1. *D is the only complex dimension of Ω with real part $D_{\mathcal{L}}$, and it is simple.*
2. *$\partial\Omega$ is Minkowski measurable.*

The above theorem applies to all self-similar strings, including the Cantor String discussed below.

Earlier, the following criterion was obtained in [24].

Theorem 1.5. *Let Ω be an arbitrary fractal string with lengths $\mathcal{L}$ and $0 < D < 1$. The following are equivalent:*

1. *$L := \lim_{j\to\infty} \ell_j \cdot j^{1/D}$ exists in $(0, \infty)$.*
2. *$\partial\Omega$ is Minkowski measurable.*

Remark 1.6. When one of the conditions of either theorem is satisfied, the Minkowski content of $\mathcal{L}$ is given by

$$\mathcal{M}(D, \mathcal{L}) = \frac{2^{1-D} L^D}{1-D}.$$

Further, under the conditions of Theorem 1.4,

$$\mathcal{M}(D, \mathcal{L}) = \operatorname{res}(\zeta_{\mathcal{L}}; D).$$

The following key result, which can be found in [26, 29], uses the complex dimensions of a fractal string in a formula for the volume of the inner ε-neighborhoods of the fractal string.

Theorem 1.7. *Under mild hypotheses, the volume of the one-sided tubular neighborhood of radius ε of the boundary of a fractal string Ω (with lengths $\mathcal{L}$) is given by the following explicit formula with error term:*

$$V(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathcal{L}}(W) \cup \{0\}} \operatorname{res}\left(\frac{\zeta_{\mathcal{L}}(s)(2\varepsilon)^{1-s}}{s(1-s)}; \omega \right) + \mathcal{R}(\varepsilon),$$

where the error term can be estimated by $\mathcal{R}(\varepsilon) = \mathcal{O}(\varepsilon^{1-\sup r})$ as $\varepsilon \to 0^+$.

Remark 1.8. In particular, in Theorem 1.7, if all the poles of $\zeta_{\mathcal{L}}$ are simple and $0 \notin \mathcal{D}_{\mathcal{L}}(W)$, then

$$V(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathcal{L}}(W)} \frac{2^{1-\omega}}{\omega(1-\omega)} \operatorname{res}(\zeta_{\mathcal{L}}; \omega) \varepsilon^{1-\omega} + \mathcal{R}(\varepsilon).$$

Remark 1.9. If $\mathcal{L}$ is a self-similar string (e.g., if its boundary is a self-similar subset of $\mathbb{R}$), then the conclusion of Theorem 1.7 holds with $\mathcal{R}(\varepsilon) \equiv 0$.

Before continuing to the section on Multifractal Analysis, consider the following six examples of fractal strings. They play important roles in many of the examples in this text.

Example 1.10 (Variants of the Cantor String).

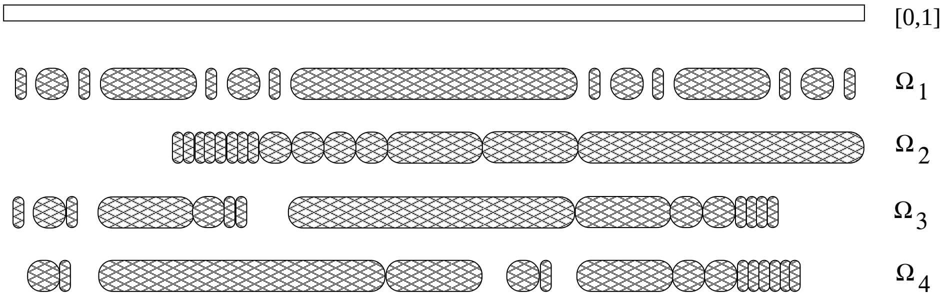

Figure 2: *Four variants of the Cantor String.*

The following four fractal strings all have the lengths of the Cantor String as their sequence of lengths $\mathcal{L}$. See Figure 2. The distinct lengths are $l_n = 3^{-n}$ with multiplicities $m_n = 2^{n-1}$ for every $n \in \mathbb{N}$. Hence, for $\mathrm{Re}s > \log_3 2$,

$$\begin{aligned}\zeta_{\mathcal{L}}(s) = \zeta_{CS}(s) &= \sum_{n=1}^{\infty} m_n l_n^s \\ &= \sum_{n=1}^{\infty} 2^{n-1} 3^{-ns} \\ &= \frac{3^{-s}}{1 - 2 \cdot 3^{-s}}.\end{aligned}$$

Upon meromorphic continuation, we see that the last equation above holds for all $s \in \mathbb{C}$, and

$$\mathcal{D}_{\mathcal{L}} = \mathcal{D}_{CS} = \left\{ \log_3 2 + \frac{2im\pi}{\log 3} \ \mid \ m \in \mathbb{Z} \right\}.$$

These fractal strings have identical geometric zeta functions and the same Minkowski dimension, namely $\log_3 2$. However, the geometric zeta functions do not detect the varied topological configurations of these strings. Throughout this text, the techniques first used in [20] are revisited to examine these differences.

Now for the construction of the four variants of the Cantor String:

Ω_1 is the Cantor String defined as the complement in $[0, 1]$ of the ternary Cantor Set. Thus $\partial\Omega_1$ is the Cantor Set itself, which is a perfect set. From Theorem 1.4, it is immediate that the Cantor Set is not Minkowski measurable. This can also be deduced from Theorem 1.5 as was first shown in [24].

Ω_2 is the fractal string that has the the same lengths as the Cantor String, but with the lengths arranged in non-increasing order from right to left. The boundary $\partial\Omega_2$ contains exactly one limit point which is at 0.

Ω_3 is a fractal string whose boundary has limit points at $0, 1$ and 3^{-n} for every $n \in \mathbb{N}$. Each point 3^{-n} is the left endpoint of an interval of length 3^{-n} and the

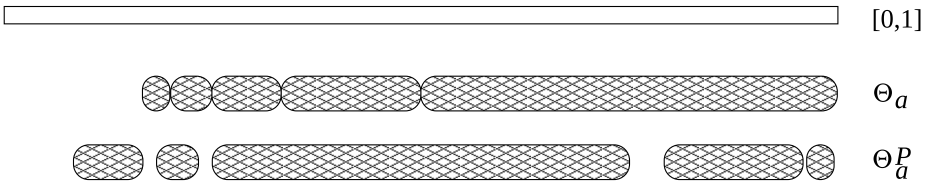

Figure 3: *Two variants of the a-String.*

remaining lengths are placed in non-increasing order from left to right. This makes the points $0, 1$ and 3^{-n} for every $n \in \mathbb{N}$ limit points of $\partial\Omega_3$.

Ω_4 is a fractal string comprised of a Cantor-like string and an isolated limit point at 1. The lengths comprising the Cantor-like string are constructed by connecting two intervals with consecutive lengths, with the remaining lengths arranged in non-increasing order from left to right, accumulating at 1. The boundary $\partial\Omega_4$ is the disjoint union of a perfect set and a collection of isolated points with a separate limit point at 1.

Example 1.11 (**Variants of the** a**-String**).

Given $a > 0$, let Θ_a be the a-String, defined as the complement in $[0, 1]$ of the countable set $\{j^{-a}\}_{j=1}^{\infty} \cup \{0\}$. That is, $\Theta_a = \bigcup_{j=1}^{\infty}((j+1)^{-a}, j^{-a})$ and therefore the boundary $\partial\Theta_a = \{j^{-a}\}_{j=1}^{\infty} \cup \{0\}$. See Figure 3. It is shown in [26, 29] that $\zeta_{\mathcal{L}_a}$ is meromorphic in $\mathbb{C}$ and that

$$\mathcal{D}_{\mathcal{L}_a} \subset \{D, D-1, D-2, ...\},$$

where $D := \frac{1}{a+1}$ is the Minkowski dimension of $\partial\Theta_a$ (or equivalently $\mathcal{L}_a$). Since it is countable, its Hausdorff dimension is 0. At once Theorem 1.4 implies that $\partial\Theta_a$ is Minkowski measurable. This can also be checked directly, as was first done in [17], or using Theorem 1.5. Further, by Remark 1.6,

$$\mathcal{M}(D, \mathcal{L}_a) = \frac{2^{1-D}a^D}{1-D} = \operatorname{res}(\zeta_{\mathcal{L}_a}; D).$$

Consider another fractal string which has the same lengths as the a-String but has perfect boundary. In particular, every boundary point is a limit point of other boundary points. To construct such a string, mimic the "middle third" interval removal construction of the usual ternary Cantor Set $\partial\Omega_1$ to get an open set Θ_a^P that has a perfect complement and the same sequence of lengths as the a-String Θ_a. For simplicity, let $a = 1$ yet still denote the corresponding open set by Θ_a^P. Recall that each $\ell_j \in \mathcal{L}_a$ has multiplicity one. See Figure 3.

The construction of Θ_a^P works as follows: Remove $l_1 = 1/2$ from the middle of the unit interval $[0, 1]$. This leaves two closed intervals of length $1/4$. Remove $l_2 = 1/6$

from the middle of the rightmost interval, leaving one interval of length 1/4 and two of length 1/24. Continue by removing each successive open interval of length l_n from the middle of the rightmost interval which is large enough to contain it with *positive* distance between this new interval and the ones which were removed before it. The limit of this process is then a perfect set, with complement Θ_a^P.

The examples presented in this section will be revisited after one of the main objects of study, the *multifractal zeta function*, is defined in Section 2. The definition was first made in [20]. Some of the results of that paper will be revisited and extended in this one.

2 Definition of the Multifractal Zeta Functions

Given $\alpha \in [-\infty, \infty]$ and $\eta \in (0,1)$, let

$$R^{\eta}(\alpha) = \bigcup_{\mathcal{R}_{\eta}(\alpha)} U.$$

For such a scale η, $R^{\eta}(\alpha)$ is a disjoint union of a finite number of intervals, each of which may be open, closed or neither and are of length at least η when $\mathcal{R}_{\eta}(\alpha)$ is non-empty. This text considers only discrete sequences of scales $\mathcal{N} = \{\eta_n\}_{n=1}^{\infty}$, with $\eta_n > 0$ for all $n \geq 1$ and the sequence strictly decreasing to zero. So for $n \in \mathbb{N}$, let

$$R^{\eta_n}(\alpha) = R^n(\alpha).$$

Further,

$$R^n(\alpha) = \bigcup_{p=1}^{r_n(\alpha)} R_p^n(\alpha),$$

where $r_n(\alpha)$ is the number of connected components $R_p^n(\alpha)$ of $R^n(\alpha)$. Denote the left and right endpoints of each interval $R_p^n(\alpha)$ by $a_R^n(\alpha, p)$ and $b_R^n(\alpha, p)$, respectively.

Given a sequence of positive real numbers $\mathcal{N} = \{\eta_n\}_{n=1}^{\infty}$ that tend to zero and a Borel measure μ on $[0,1]$, one can examine the way μ changes with respect to a fixed regularity α between stages $n-1$ and n. Consider the symmetric difference, denoted $\ominus$, between $R^{n-1}(\alpha)$ and $R^n(\alpha)$. Recall that for two sets A and B, their symmetric difference $A \ominus B$ is given by $A \ominus B = (A \setminus B) \cup (B \setminus A)$. Let

$$J^1(\alpha) = R^1(\alpha),$$

and for $n \geq 2$, let

$$J^n(\alpha) = R^{n-1}(\alpha) \ominus R^n(\alpha).$$

For all $n \in \mathbb{N}$, $J^n(\alpha)$ is also a disjoint union of intervals $J^n_p(\alpha)$, each of which may be open, closed, or neither. Then

$$J^n(\alpha) = \bigcup_{p=1}^{j_n(\alpha)} J^n_p(\alpha),$$

where $j_n(\alpha)$ is the number of connected components $J^n_p(\alpha)$ of $J^n(\alpha)$. The left and right endpoints of each interval $J^n_p(\alpha)$ are denoted by $a^n_J(\alpha, p)$ and $b^n_J(\alpha, p)$, respectively.

For a given regularity $\alpha \in [-\infty, \infty]$ and a measure μ, the sequence $\mathcal{N}$ determines another sequence corresponding to the lengths of the connected components of the $J^n(\alpha)$. That is, the $J^n(\alpha)$ describe the way μ behaves between scales η_{n-1} and η_n with respect to α. However, there is some redundancy with this set-up. Indeed, a particular regularity value may occur at all scales smaller than a certain fixed scale in the same location (in the specific sense defined below). The desire to eliminate this redundancy will be clarified with some examples below. The next step is introduced to carry out this elimination.

Let $K^1(\alpha) = J^1(\alpha) = R^1(\alpha)$. For $n \geq 2$, let $K^n(\alpha)$ be the union of the subcollection of intervals in $J^n(\alpha)$ comprised of the intervals that have left and right endpoints distinct from, respectively, the left and right endpoints of the intervals in $R^{n-1}(\alpha)$. Thus,

$$K^n(\alpha) = \bigcup_{p=1}^{k_n(\alpha)} K^n_p(\alpha) \subset J^n(\alpha),$$

where $k_n(\alpha)$ is the number of connected components $K^n_p(\alpha)$ of $K^n(\alpha)$. That is, the $K^n_p(\alpha)$ are the $J^n_p(\alpha)$ such that $a^n_J(\alpha, p_1) \neq a^{n-1}_R(\alpha, p_2)$ and $b^n_J(\alpha, p_1) \neq b^{n-1}_R(\alpha, p_2)$ for all $p_1 \in \{1, ..., j_n(\alpha)\}$ and $p_2 \in \{1, ..., r_n(\alpha)\}$.

Collecting the lengths of the intervals $K^n_p(\alpha)$ allows one to define a zeta function without specifying an open set. Let

$$\mathcal{K}^\mu_\mathcal{N}(\alpha) = \{|K^n_p(\alpha)| \mid n \in \mathbb{N},\ p \in \{1, ..., k_n(\alpha)\}\}.$$

To define a generalization of the geometric zeta function of a fractal string, consider a family of zeta functions parameterized by the regularity values of the measure μ.

Definition 2.1. *The* <u>*multifractal zeta function*</u> *of a measure μ, sequence $\mathcal{N}$ and with associated regularity value $\alpha \in [-\infty, \infty]$ is*

$$\zeta^\mu_\mathcal{N}(\alpha, s) = \sum_{n=1}^{\infty} \sum_{p=1}^{k_n(\alpha)} |K^n_p(\alpha)|^s,$$

for Res *large enough.*

If it is assumed that, as a function of $s \in \mathbb{C}$, $\zeta_{\mathcal{N}}^{\mu}(\alpha, s)$ admits a meromorphic continuation to an open neighborhood of a window W, then one may also consider the poles of these zeta functions, as in the case of the complex dimensions of a fractal string (see Section 1).

Definition 2.2. *For a measure μ, sequence $\mathcal{N}$ which decreases to zero and regularity value α, the set of complex dimensions with parameter α is*

$$\mathcal{D}_{\mathcal{N}}^{\mu}(\alpha, W) = \{\omega \in W \mid \zeta_{\mathcal{N}}^{\mu}(\alpha, s) \text{ has a pole at } \omega\}.$$

When $W = \mathbb{C}$, simply write $\mathcal{D}_{\mathcal{N}}^{\mu}(\alpha)$. The following section describes how the multifractal zeta function with regularity $\alpha = \infty$ precisely recovers the geometric zeta function.

3 Regularity Value Infinity and Geometric Zeta Functions

The goal of this section is to show that regularity value $\alpha = \infty$ yields the geometric zeta function of the complement of the support of a given positive Borel measure μ on $[0,1]$. Let E^c denote the complement of E in $[0,1]$ (that is, $E^c := [0,1]\backslash E$.) Since $(supp(\mu))^c$ is open, it can be considered as a *fractal string* Ω_μ whose lengths $\mathcal{L}_\mu$ are those of the disjoint intervals (a_j, b_j) where $(supp(\mu))^c = \Omega_\mu = \cup_{j=1}^{\infty}(a_j, b_j)$. Let $\{\ell_j\}_{j=1}^{\infty}$ be the lengths of $\mathcal{L}_\mu$ and let $\{l_n\}_{n=1}^{\infty}$ be the distinct lengths of $\mathcal{L}_\mu$ with multiplicities $\{m_n\}_{n=1}^{\infty}$.

The following technical lemma is used in the proof of the theorem below.

Lemma 3.1. *Suppose* $\{x\} = supp(\mu) \cap U$ *for some* $U \in \mathbf{X}([0,1])$. *Then*

$$A(U) = \infty \Leftrightarrow \sigma(\{x\}) = 0.$$

Proof. Clearly, $\mu(\{x\}) \neq 0 \Leftrightarrow \mu(U) > |U| \Leftrightarrow A(U) \neq \infty$. □

The lemma implies that a closed interval with regularity ∞ can have non-empty intersection with $supp(\mu)$ only if the intersection contains at most two points that carry no weight. It must also be true that if $a_j = b_k = x$, then $\mu(\{x\}) \neq 0$, otherwise the complement of $supp(\mu)$ would break down into a different union of open intervals, giving rise to a different fractal string. See Figures 4 and 5 for an illustration of the construction of the first several terms of the multifractal zeta function that appears in Theorem 3.2, applied to a measure which supported on the Cantor Set. Specifically, Figure 5 shows the construction of the multifractal zeta function $\zeta_{\mathcal{N}}^{\mu_1}(\infty, s)$, where the measure μ_1 has its singular part supported on the endpoints of the Cantor String.

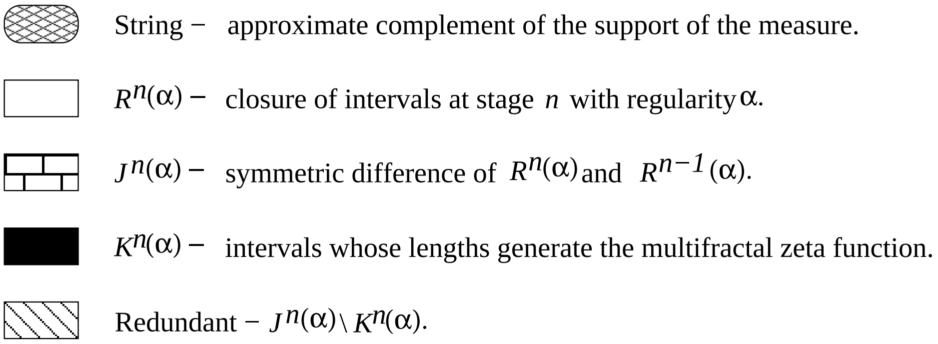

Figure 4: *Key for the construction of multifractal zeta functions.*

The chosen sequence of scales is $\mathcal{N} = \{3^{-n-1}\}_{n=1}^{\infty}$. The solid black bars represent the lengths used to construct $\zeta_{\mathcal{N}}^{\mu_1}(\infty, s)$, which is equal to the geometric zeta function of the Cantor String.

The following theorem is completely analogous to one of the main theorems of [20], Theorem 5.2. Here regularity value ∞ is related to the support of a measure μ, whereas in [20] the measures under consideration are taken to be singular with respect to the Lebesgue measure and the pertinent regularity value is 1. Aside from these differences, the result and the proof are the same.

Theorem 3.2. *The multifractal zeta function of a positive Borel measure μ, any sequence $\mathcal{N}$ such that $\eta_n \searrow 0$ and regularity $\alpha = \infty$ is the geometric zeta function of $(supp(\mu))^c$. That is,*

$$\zeta_{\mathcal{N}}^{\mu}(\infty, s) = \zeta_{\mathcal{L}_\mu}(s).$$

Proof. Recall the notation introduced at the beginning of Section 2. For all $n \in \mathbb{N}$,

$$U \in \mathcal{R}_{\eta_n}(\infty) \Leftrightarrow A(U) = \infty \text{ and } |U| = \eta_n.$$

Further,

$$\infty = A(U) = \frac{\log(\mu(U))}{\log|U|}$$

Therefore, $\forall n \in \mathbb{N}, U \in \mathcal{R}_{\eta_n}(\infty)$ only if $\mu(U) = 0$.

The sets $\mathcal{R}_{\eta_n}(\infty)$ depend further upon whether any of the endpoints of the intervals $I_j = (a_j, b_j)$ which comprise $\Omega_\mu = (supp(\mu))^c$ contain mass as singletons. If $\mu(\{a_j\}) \neq 0$ and $\mu(\{b_j\}) \neq 0$ for all $j \in \mathbb{N}$, then

$$R^n(\infty) = \bigcup_{\ell_j > \eta_n} I_j \subset \Omega_\mu.$$

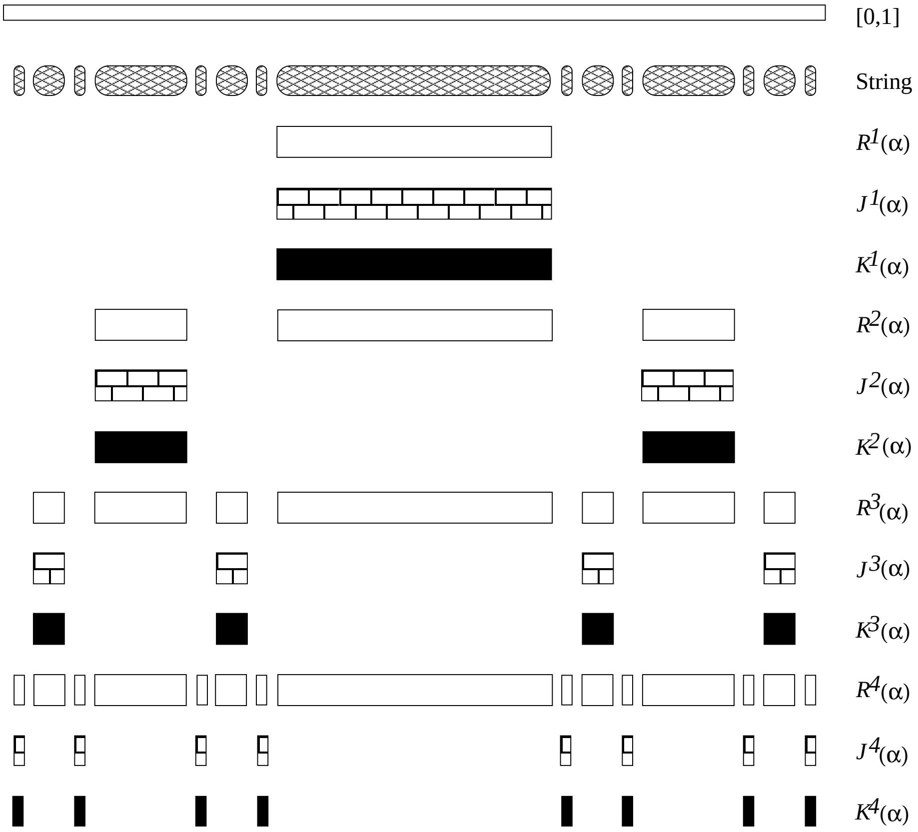

Figure 5: *Construction of* $\zeta_{\mathcal{N}}^{\mu_1}(\infty, s)$.

If, for instance, $\mu(\{a_j\}) = 0$ for some j, then $R^n(\infty)$ contains a_j for all n such that $\ell_j > \eta_n$. If $\mu(\{a_j\}) = \mu(\{b_j\}) = 0$, then $R^n(\infty)$ contains a_j and b_j for all n such that $\ell_j \geq \eta_n$. Including these endpoints does not affect the sequence of lengths that define the multifractal zeta function. The theorem holds if $\mu(\{a_j\}) = 0$ or $\mu(\{b_j\}) = 0$ (or both) for some (or all) $j \in \mathbb{N}$. The proof below would change slightly, however the lengths l_n and multiplicities m_n which constitute the multifractal zeta function would not. Essentially, at most a countably infinite number of singletons would be added to the collection of the $R^n(\infty)$ in such a way that the lengths ultimately remain the same. This is because the disjoint components of the $R^n(\infty)$ are separated by a singleton with mass or a positive distance, as follows from Lemma 3.1. For instance, if $\mu(\{a_j\}) = \mu(\{b_j\}) = 0$ and $\eta_n \neq \ell_j$ for all $n, j \in \mathbb{N}$, then

$$R^n(\infty) = \overline{\bigcup_{\ell_j > \eta_n} I_j}.$$

If each ℓ_j is equal to some η_n, then

$$R^n(\infty) = \overline{\bigcup_{\ell_j \geq \eta_n} I_j}.$$

In other words, if an interval of length ℓ_j is not picked up at stage n, it will be picked up at some later stage. The case where every endpoint contains mass is proven here, without loss of generality.

Suppose $\mu(\{a_j\}) \neq 0$ and $\mu(\{b_j\}) \neq 0$ for all $j \in \mathbb{N}$. Then $R^n(\infty) = \bigcup_{\ell_j > \eta_n} I_j$ implies that, for $n \geq 2$,

$$\begin{aligned} J^n(\infty) &= \left(\bigcup_{\ell_j > \eta_{n-1}} I_j \right) \ominus \left(\bigcup_{\ell_j > \eta_n} I_j \right) \\ &= \left(\bigcup_{\ell_j > \eta_n} I_j \right) \setminus \left(\bigcup_{\ell_j > \eta_{n-1}} I_j \right) \\ &= \bigcup_{\eta_{n-1} \geq \ell_j > \eta_n} I_j. \end{aligned}$$

Since $R^{n-1}(\infty) \subset R^n(\infty)$ for all $n \geq 2$, the intervals $J^n(\infty)$ have no redundant lengths. That is, $a_J^n(\infty, p_1) \neq a_R^{n-1}(\infty, p_2)$ and $b_J^n(\infty, p_1) \neq b_R^{n-1}(\infty, p_2)$ for all $n \geq 2$ and $p_1, p_2 \in \{1, ..., j_n(\infty)\}$. This implies

$$K^n(\infty) = J^n(\infty) = \bigcup_{\eta_{n-1} \geq \ell_j > \eta_n} I_j.$$

Furthermore,

$$|K^n(\infty)| = \sum_{p=1}^{k_n(\infty)} |K_p^n(\infty)| = \sum \ell_j,$$

where the last sum is taken over all j such that $\eta_{n-1} \geq \ell_j > \eta_n$. Since $\eta_n \searrow 0$, each length ℓ_j is eventually picked up. Therefore,

$$\begin{aligned}
\zeta_{\mathcal{N}}^{\mu}(\infty, s) &= \sum_{n=1}^{\infty} \sum_{p=1}^{k_n(\infty)} |K_p^n(\infty)|^s \\
&= \sum_{n=1}^{\infty} \sum \ell_j^s \\
&= \sum_{n=1}^{\infty} m_n l_n^s \\
&= \zeta_{\mathcal{L}_\mu}(s).
\end{aligned}$$

□

Corollary 3.3. *Under the assumptions of Theorem 3.2, the complex dimensions of the fractal string Ω_μ coincide with the poles of the multifractal zeta function $\zeta_{\mathcal{N}}^{\mu}(\infty, s)$. That is,*

$$\mathcal{D}_{\mathcal{N}}^{\mu}(\infty, W) = \mathcal{D}_{\mathcal{L}_\mu}(W)$$

for every window W.

The key in Figure 4 will be used for the examples that analyze the fractal strings below. Figure 5 shows the first four steps in the construction of a multifractal zeta function with regularity ∞ for a measure supported on the Cantor Set.

In the case of $\partial\Omega_\mu = supp(\mu)$, the abscissa of convergence of $\zeta_{\mathcal{N}}^{\mu}(\infty, s)$, $D_{\mathcal{L}_\mu}$, is the Minkowski dimension of $supp(\mu)$. Note that the choice of sequence of scales $\mathcal{N}$ does not affect the result of Theorem 3.2, as long as the sequence decreases to zero. Further, since the geometric zeta function of the fractal string $\Omega_\mu = (supp(\mu))^c$ is recovered for regularity value $\alpha = \infty$, the result of Theorem 3.2 depends only on the lengths $\mathcal{L}_\mu$ of the open intervals whose disjoint union is Ω_μ and not on their topological configuration. This is not the case for other regularity values. Specifically, the choice of sequence of scales $\mathcal{N}$ and the topological configuration of Ω_μ greatly affect the form of the multifractal zeta functions for other regularity values. The cases $\alpha = 0$ and $-\infty$ are examined in the following sections.

4 Effective Strings and Effective Lengths

In this section, the fractal string Ω is taken as an open subset of $[0,1]$ which is comprised of countably many open intervals such that $|\Omega| = 1$ and $\partial\Omega = [0,1]\backslash\Omega$. Associate to Ω its sequence of lengths $\mathcal{L}$. For such Ω, the endpoints are dense in $\partial\Omega$. Indeed, if there were a point in $\partial\Omega$ away from any endpoint, then it would be away from Ω itself. This allows us to define, in a natural way, measures with a countable number of point-masses contained in the boundary of Ω. Let

$$\mu_\Omega = \sum_{j=1}^{\infty} (\delta_{a_j} + \delta_{b_j}),$$

where the (a_j, b_j) are the open intervals whose disjoint union is Ω and the sum is taken so that if $a_{j_1} = x = b_{j_2}$, the point has mass 1. This notation will be used repeatedly throughout the text to facilitate use of multifractal zeta functions for the investigation of geometric and topological properties of fractal strings.

Regularity values $-\infty, 0$ and ∞ yield interesting results for a fractal string Ω when the measure μ_Ω is taken under consideration. For $\alpha = \infty$, $\mathcal{R}_{\eta_n}(\infty)$ is the collection of closed intervals of length η_n which contain no mass. This results in a multifractal zeta function equal to the geometric zeta function of the fractal string Ω, as in Theorem 3.2. This result is independent of the choice of scales $\mathcal{N}$.

For $\alpha = -\infty$, $\mathcal{R}_{\eta_n}(-\infty)$ is the collection of closed intervals of length η_n which contain infinitely many point-masses. In other words, $\mathcal{R}_{\eta_n}(-\infty)$ is the collection of closed intervals of length η_n that contain a neighborhood of a limit point of the endpoints of Ω. This section investigates the behavior of the resulting multifractal zeta function, which depends on $\mathcal{N}$.

Regularity $\alpha = 0$ corresponds to those closed intervals which contain exactly one endpoint of a fractal string Ω. This regularity value is considered in a few examples in the next section, along with examples involving $\alpha = -\infty$.

Defined below is a special fractal string, the *effective string*, which describes the collection of limit points of the boundary $\partial\Omega$ of a fractal string Ω. This fractal string is related to the sequence of *effective lengths* from [20] and provides an upper bound on the abscissae of convergence of the multifractal zeta functions with regularity $-\infty$.

Definition 4.1. *For a given fractal string Ω, the effective string Ω_E is the fractal string contained in the unit interval whose boundary points are* $0, 1$, *and the collection of limit points of $\partial\Omega$.*

The lengths of Ω_E are the *gap lengths* from [20]. The distinct gap lengths are the distinct sums $g_k := \sum \ell_j$ where $k \in \mathbb{N}$ and the sums are taken over all j's such that the disjoint subintervals (a_j, b_j) of Ω are adjacent and have rightmost and/or leftmost endpoints (or limits thereof) which are $0, 1$ or limit points of $\partial\Omega$.

The definition of effective string is motivated by a key property of the Hausdorff dimension $\dim_H$: it is countably stable, that is,

$$\dim_H(\cup_{n=1}^{\infty} A_n) = \sup_{n \geq 1} \left(\dim_H(A_n)\right).$$

For this and other properties of $\dim_H$, see [8]. Consequently, countable sets have Hausdorff dimension zero and countable collections of isolated points do not contribute to the Hausdorff dimension of a given set. A relation between the Minkowski dimension of $\partial\Omega_E$ and the Hausdorff dimension of $\partial\Omega$ is considered below in Corollary 4.5. The effective string is precisely the fractal string that ignores the isolated endpoints of the fractal string Ω. As a consequence, in certain cases the Minkowski dimension of the boundary of the effective string $\partial\Omega_E$ coincides with the Hausdorff dimension of $\partial\Omega$.

Before stating Corollary 4.5, it is necessary to define some notions regarding sets which are known as *self-similar* and recall a well-known result for such sets. The construction of this definition and result can be found in [8], page 117. A *similarity* S_j is a map from $[0, 1]$ to $[0, 1]$ such that $|S_j(x) - S_j(y)| = c_j|x - y|$, for all $x, y \in [0, 1]$ and where $0 < c_j < 1$.

Definition 4.2. *For a finite collection of similarities $\{S_1, ..., S_k\}$, the set F which is invariant under this collection is called self-similar. That is, F is the unique invariant set that satisfies*

$$F = \bigcup_{j=1}^{k} S_j(F).$$

Theorem 4.4 and its corollary require one further condition on the collection of similarities; the *open set condition.*

Definition 4.3. *A collection of similarities* $\{S_1, ..., S_k\}$ *satisfies the open set condition if there is a non-empty open set* $V \subset [0,1]$ *such that*

$$V \supset \bigcup_{j=1}^{k} S_j(V),$$

where the union is disjoint.

The full statement and proof of Theorem 4.4 can be found on page 118 of [8].

Theorem 4.4. *Suppose similarities* $S_1, ..., S_k$ *satisfy the open set condition with invariant set* F. *Then*

$$\dim_H(F) = \dim_M(F).$$

Corollary 4.5. *Suppose a fractal string* $\Omega \subset [0,1]$ *is such that* $|\Omega| = 1$, $\partial\Omega = [0,1]\backslash\Omega$ *and* $\partial\Omega$ *consists of a disjoint union of:*

1. *A self-similar set which is the invariant set of a collection of similarities* $\{S_1, ..., S_k\}$.
2. *At-most countably many isolated points.*
3. *A finite collection of limit points which are positive distances away from the self-similar set.*

Then the Minkowski dimension of $\partial\Omega_E$ *is equal to the Hausdorff dimension of* $\partial\Omega$. *That is,*

$$\dim_H(\partial\Omega) = \dim_M(\partial\Omega_E).$$

Proof. Case (i): $\partial\Omega$ has finitely many limit points.

In this case, $\partial\Omega$ consists of countably many points and $\partial\Omega_E$ is finite. Thus, the countable stability of the Hausdorff dimension and the finite stability of the Minkowski dimension immediately yield

$$\dim_H(\partial\Omega) = 0 = \dim_M(\partial\Omega_E).$$

Case (ii): $\partial\Omega$ consists of a self-similar set, isolated points and a finite number of limit points which are disjoint from the self-similar set.

This case follows from Theorem 4.4 and two facts: First, the countable collection of isolated points and finite limit points away from the self-similar set do not contribute to the Hausdorff dimension of $\partial\Omega$. Second, the finite collection of separate limit points contributes a finite number of terms to the geometric zeta function of the effective string Ω_E, thus it does not affect the abscissa of convergence and hence the Minkowski dimension of $\partial\Omega_E$. As a result, only the self-similar subset of $\partial\Omega$ contributes to the values of $\dim_H(\partial\Omega)$ and $\dim_M(\partial\Omega_E)$, which are equal by Theorem 4.4. $\square$

Remark 4.6. Corollary 4.5 applies to the fractal strings Ω_1, Ω_2 and Ω_4. Computations can be found in Section 5. The fractal string Ω_3 does not satisfy the conditions of Corollary 4.5, but the result holds. That is, $\dim_H(\partial\Omega_3) = \dim_M(\partial\Omega_{3,E})$. However, for the fractal string Ω_γ in Section 7, which is quite similar to Ω_3, the relation is $\dim_H(\partial\Omega_\gamma) < \dim_M(\partial\Omega_{\gamma,E})$. Still, if the effective string of the effective string is considered, then $\dim_H(\partial\Omega_\gamma) = \dim_M(\partial\Omega_{\gamma,E,E})$.

The multifractal zeta function of the measure μ_Ω with regularity $-\infty$ detect the gaps which define the effective string Ω_E and describe their structure as $\eta_n \to 0$. The following definition is from [20].

Definition 4.7. *The sequence of* <u>*effective lengths*</u> *of a fractal string* Ω *with respect to the sequence* $\mathcal{N}$ *is*

$$\mathcal{K}_{\mathcal{N}}^{\mu_\Omega}(-\infty) := \{|K_p^n(-\infty)| \mid n \in \mathbb{N},\ p \in \{1, ..., k_n(-\infty)\}\},$$

where $\mu_\Omega := \sum_{j=1}^{\infty}(\delta_{a_j} + \delta_{b_j})$.

Regularity $-\infty$ picks up closed intervals of all sizes $\eta_n \in \mathcal{N}$ that contain an open neighborhood of a limit point of $\partial\Omega$. The effective lengths (and hence their multifractal zeta function) describe the gaps between these accumulation points as detected at all scales $\eta_n \in \mathcal{N}$.

The effective lengths have the following description: For the scale η_1, $K^1(-\infty)$ is the union of the collection of connected components of $R^1(-\infty)$. For η_n such that $n \geq 2$,

$$|K_p^n(-\infty)| = g_k - \eta_n$$

if η_n is the scale that first detects the gap g_k, that is, if η_n is the unique first scale η_k^E such that

$$2\eta_{n-1} > g_k \geq 2\eta_n.$$

Under appropriate re-indexing, the effective lengths with multiplicities $\{m_{E,k}\}_{k\geq 2}$ are $\{l_{E,k}\}_{k\geq 2}$, given by

$$l_{E,k} := g_k - 2\eta_k^E,$$

where the gaps g_k are those such that $2\eta_1 > g_k$ and the $\eta_k^E \in \mathcal{N}_E \subset \mathcal{N}$ are the effective scales with respect to $\mathcal{N}$ that detect these gaps. The result is summarized in the next theorem, which gives a formula for the multifractal zeta function of the measure μ_Ω with sequence of scales $\mathcal{N}$ at regularity $-\infty$. The second formula in Theorem 5.1 below can be viewed as a corollary to this theorem.

Theorem 4.8. *Given a fractal string* Ω*, the measure* μ_Ω *and a sequence of scales* $\mathcal{N}$ *such that* $\eta_n \searrow 0$ *yield the following multifractal zeta function at regularity* $-\infty$*:*

$$\zeta_{\mathcal{N}}^{\mu_\Omega}(-\infty, s) \quad = \quad \sum_{p=1}^{k_1(-\infty)} |K_p^1(-\infty)|^s + \sum_{k=1}^{\infty} m_{E,k} l_{E,k}^s$$

for Res *large enough.*

Remark 4.9. Though the effective lengths are not the lengths of the effective string, they are related as in the following corollary.

Corollary 4.10. *For a given fractal string* Ω*, the Minkowski dimension of* $\partial\Omega_E$ *is an upper bound of the collection of abscissae of convergence of the multifractal zeta functions* $\zeta_{\mathcal{N}}^{\mu_\Omega}(-\infty, s)$ *for any choice of scales* $\mathcal{N}$*.*

Proof. The proof follows immediately from the fact that adding an entire function to a meromorphic function does not change its poles and that the gap lengths dominate the effective lengths (that is, $g_k > g_k - \eta_n = l_{E,k}$). ☐

5 Regularity Values Zero and Negative Infinity

Let us a revisit one of the results of [20]. If the open set Ω has a perfect boundary there is a relatively simple breakdown of all possible multifractal zeta functions for the measure μ_Ω. Recall that a set is perfect if it is equal to its set of limit points. Further, note that the boundary of a fractal string is closed; hence, it is perfect if and only if it does not have isolated points. The simplicity of the breakdown of the multifractal zeta functions for perfect strings is due to the fact that every point-mass is a limit point of other point-masses. Consequently, the only α that do not yield identically zero multifractal zeta functions are $-\infty, 0, \infty$ and those which correspond to a length of $\mathcal{N}$ and two point-masses. A result very similar to the following theorem was first obtained in [20] where, in a slightly different setting, the regularity values $\alpha = 1$ and $-\infty$ yield multifractal zeta functions with poles.

Theorem 5.1. *Let $\Omega = \bigcup_{j=1}^{\infty}(a_j, b_j)$ be a fractal string with sequence of lengths $\mathcal{L}$ and perfect boundary. Suppose that $\mathcal{N}$ is a sequence such that $l_n > \eta_n \geq l_{n+1}$ and $l_n > 2\eta_n$, for all $n \in \mathbb{N}$. Then*

$$\begin{aligned}
\zeta_{\mathcal{N}}^{\mu_\Omega}(\infty, s) &= \zeta_{\mathcal{L}}(s) = \sum_{n=1}^{\infty} m_n l_n^s, \\
\zeta_{\mathcal{N}}^{\mu_\Omega}(0, s) &= \sum_{n=1}^{\infty} 2 m_n \eta_n^s, \ \textit{and} \\
\zeta_{\mathcal{N}}^{\mu_\Omega}(-\infty, s) &= h(s) + \sum_{n=2}^{\infty} m_n (l_n - 2\eta_n)^s,
\end{aligned}$$

where $h(s)$ is the entire function given by $h(s) = \sum_{p=1}^{k_1(-\infty)} |K_p^1(-\infty)|^s$. For every other real number α, $\zeta_{\mathcal{N}}^{\mu_\Omega}(\alpha, s)$ is entire.

Proof. $\zeta_{\mathcal{N}}^{\mu_\Omega}(\infty, s) = \zeta_{\mathcal{L}}(s)$ holds by Theorem 3.2.

Since $l_n > 2\eta_n$, we have

$$R^n(-\infty) = (\bigcup_{\ell_j > \eta_n} [a_j + \eta_n, b_j - \eta_n])^c.$$

For $n \geq 2$, $J^n(-\infty)$ is made up of m_n intervals of length $l_n - 2\eta_n$ and $2\sum_{p=1}^{n-1} m_p$ intervals of length $\eta_{n-1} - \eta_n$. That is, at each stage $n \geq 2$, we pick up two $\eta_{n-1} - \eta_n$ terms for each $\ell_j \geq l_{n-1}$ from the previous stage and one $l_n - 2\eta_n$ term for each $\ell_j = l_n$. By construction, the sets $K^n(-\infty)$ do not include the redundant $\eta_{n-1} - \eta_n$ terms. Therefore,

$$\zeta_{\mathcal{N}}^{\mu_\Omega}(-\infty, s) = \sum_{p=1}^{k_1(-\infty)} |K_p^1(-\infty)|^s + \sum_{n=2}^{\infty} m_n (l_n - 2\eta_n)^s.$$

For $\alpha = 0$,

$$R^n(0) = \bigcup_{\ell_j > \eta_n} ([a_j, a_j + \eta_n] \cup [b_j - \eta_n, b_j])$$

for all $n \geq 1$. Thus, $R^n(0) \subset R^{n+1}(0)$ for all $n \geq 1$. For $n \geq 2$, the $J^n(0)$ contain $\sum_{p=1}^{n-1} 2m_p$ redundant lengths of the form $(a_j + \eta_n, a_j + \eta_{n-1}]$ or $[b_j - \eta_{n-1}, b_j - \eta_n)$. The remaining $2m_n$ lengths of the form $[a_j, a_j + \eta_n]$ or $[b_j - \eta_n, b_j]$ constitute the $K^n(0)$, which in turn constitute the multifractal zeta function. That is,

$$\zeta_{\mathcal{N}}^{\mu_\Omega}(0, s) = \sum_{n=1}^{\infty} 2 m_n \eta_n^s.$$

The remaining types of multifractal zeta functions are either identically zero or a finite sum of terms of the form η_n^s if the regularity is of the form $\alpha = \log 2 / \log \eta_n$, which is strictly increasing as $n \to \infty$. $\square$

For certain fractal strings with perfect boundaries and a naturally chosen sequence, Theorem 5.1 has the following corollary.

Corollary 5.2. *For a fractal string Ω with perfect boundary, total length 1, lengths $\mathcal{L}$ given by $l_n = ca^{-n}$ with multiplicities m_n such that $a > 2$ and c is a positive normalization constant, and given a sequence of scales $\mathcal{N}$ where $\eta_n = l_{n+1} = ca^{-n-1}$,*

$$\zeta_{\mathcal{N}}^{\mu_\Omega}(-\infty, s) = f_0(s) + f_1(s)\zeta_{\mathcal{L}}(s),$$

where $f_0(s)$ and $f_1(s)$ are entire. Further,

$$\zeta_{\mathcal{N}}^{\mu_\Omega}(0, s) = 2a^{-s}\zeta_{\mathcal{L}}(s).$$

Proof. By Theorem 5.1,

$$\begin{aligned}
\zeta_{\mathcal{N}}^{\mu_\Omega}(-\infty, s) &= h(s) + \sum_{n=2}^{\infty} m_n (l_n - 2l_{n+1})^s \\
&= h(s) + c^s \sum_{n=2}^{\infty} m_n (a^{-n} - 2a^{-n-1})^s \\
&= h(s) + c^s \sum_{n=2}^{\infty} m_n \left(\frac{a-2}{a^{n+1}}\right)^s \\
&= h(s) + c^s \left(\frac{a-2}{a}\right)^s \sum_{n=2}^{\infty} m_n a^{-ns} \\
&= h(s) + c^s \left(\frac{a-2}{a}\right)^s \left(\zeta_{\mathcal{L}}(s) - m_1 a^{-s}\right).
\end{aligned}$$

Therefore, the result holds with

$$f_0(s) := h(s) - m_1 c^s \left(\frac{a-2}{a^2}\right)^s$$

and

$$f_1(s) := c^s \left(\frac{a-2}{a}\right)^s.$$

The second formula follows immediately from Theorem 5.1. □

Corollary 5.2 clearly indicates that, in general, $\zeta_{\mathcal{N}}^{\mu_\Omega}(-\infty, s)$ may have poles.

Corollary 5.3. *With the same conditions as Corollary 5.2, the complex dimensions with parameters* $-\infty, 0$ *and* ∞ *coincide with the (usual) complex dimensions of the fractal string* Ω. *Specifically,*

$$\mathcal{D}_{\mathcal{N}}^{\mu}(-\infty, W) = \mathcal{D}_{\mathcal{N}}^{\mu}(0, W) = \mathcal{D}_{\mathcal{N}}^{\mu}(\infty, W) = \mathcal{D}_{\mathcal{L}}(W).$$

Example 5.4 (Regularity $-\infty$ and Variants of the Cantor String).

Let $\mathcal{L}$ be the sequence of lengths in the complement of the Cantor Set, which is also known as the Cantor String. That is, $l_n = 3^{-n}$ and $m_n = 2^{n-1}$, as in Section 1. We will discuss several examples involving this sequence of lengths, specifically Ω_1–Ω_4 from Section 1.

Consider the following measures which have singularities on a portion of the boundary of the Ω_q for $q = 1, 2, 3, 4$:

$$\mu_q = \mu_{\Omega_q}.$$

These measures have a unit point-mass at every endpoint of the intervals which comprise the respective Ω_q.

Let $\mathcal{N}$ be such that $l_n > \eta_n \geq l_{n+1}$ and $l_n > 2\eta_n$. Such sequences exist for the Cantor String. For instance, $\forall n \in \mathbb{N}$, let $\eta_n = l_{n+1} = 3^{-n-1}$. Theorem 3.2 for $q = 1, 2, 3, 4$ yields

$$\zeta_{\mathcal{N}}^{\mu_q}(\infty, s) = \zeta_{CS}(s) = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}}.$$

For $\alpha = -\infty$ and this $\mathcal{N}$, the multifractal zeta functions for these measures differ because they are sensitive to the topological configuration of the fractal strings, as described in the previous section. In these cases,

$$-\infty = A(U) \quad = \quad \frac{\log(\sum_{a_j, b_j \in U} 1)}{\log |U|}$$

if and only if

$$\#\{j \mid a_j \in U\} + \#\{j \mid b_j \in U\} = \infty.$$

All of the Ω_q have the same sequence of lengths, so they all have the same Minkowski dimension, namely $\log_3 2$. However, their respective Hausdorff dimensions do not necessarily coincide. Certain multifractal zeta functions detect this difference but the theory of fractal strings developed in [26, 29] does not. For a natural choice of sequence of scales $\mathcal{N}$ and regularity $-\infty$, some of the multifractal zeta functions of the measures $\mu_q = \mu_{\Omega_q}$ (as above) have poles on a discrete line above and below the Hausdorff dimension of the boundaries of these fractal strings (see Figures 6–9). In [26, 29] it is shown that the set of *complex dimensions* for the fractal strings Ω_q, for $q = 1, 2, 3, 4$, is

$$\mathcal{D}_{\mathcal{N}}^{\mu_q}(\infty) = \mathcal{D}_{CS} = \left\{ \log_3 2 + \frac{2i\pi m}{\log 3} \mid m \in \mathbb{Z} \right\}.$$

The geometric zeta function of the Cantor String does not see any difference between the open sets Ω_q. However, the multifractal zeta functions of the measures μ_q with the same such $\mathcal{N}$ and regularity $\alpha = -\infty$ are quite different.

The Hausdorff dimension of the boundaries of each fractal string Ω_q are easily determined. For a set F, denote the Hausdorff dimension by $\dim_H(F)$ and the Minkowski dimension by $\dim_M(F)$. For $q = 1, 2, 3, 4$,

$$\begin{aligned} \dim_H(\partial\Omega_1) &= \dim_M(\partial\Omega_q) = \log_3 2, \\ \dim_H(\partial\Omega_2) &= \dim_H(\partial\Omega_3) = 0, \\ \dim_H(\partial\Omega_4) &= \log_9 2. \end{aligned}$$

The first line holds because Minkowski dimension depends only on the lengths of the fractal strings. Also, the Cantor Set $\partial\Omega_1$ is a strictly self-similar set whose similarities satisfy the *open set condition*, as defined in [8] and Section 4. Thus, the Minkowski and Hausdorff dimensions coincide for $\partial\Omega_1$. The second line holds because $\partial\Omega_2$ and $\partial\Omega_3$ are countable sets. The third line holds because Hausdorff dimension is (countably) stable and $\partial\Omega_4$ is the disjoint union of a strictly self-similar set and a countable set.

Theorem 5.1, Corollary 5.2 and Theorem 4.8 will be used to generate the following closed forms of the zeta functions $\zeta_{\mathcal{N}}^{\mu_q}(-\infty, s)$.

For the Cantor String Ω_1 and the corresponding measure μ_1, we have by Corollary 5.2,

$$\begin{aligned}
\zeta_{\mathcal{N}}^{\mu_1}(-\infty, s) &= 2\left(\frac{1}{3}+\frac{1}{9}\right)^s + \sum_{n=2}^{\infty} 2^{n-1}\left(\frac{1}{3^n}-\frac{2}{3^{n+1}}\right)^s \\
&= 2\left(\frac{4}{9}\right)^s + \sum_{n=2}^{\infty} 2^{n-1}\left(\frac{1}{3^{n+1}}\right)^s \\
&= 2\left(\frac{4}{9}\right)^s + \frac{2}{27^s}\sum_{n=0}^{\infty} 2^n 3^{-ns} \\
&= 2\left(\frac{4}{9}\right)^s + \frac{2}{27^s}\left(\frac{1}{1-2\cdot 3^{-s}}\right).
\end{aligned}$$

See Figure 6. The poles of $\zeta_{\mathcal{N}}^{\mu_1}(-\infty, s)$ are the same as the poles of the geometric zeta function of the Cantor String. They are given by

$$\mathcal{D}_{\mathcal{N}}^{\mu_1}(-\infty) = \left\{\log_3 2 + \frac{2i\pi m}{\log 3} \mid m \in \mathbb{Z}\right\} = \mathcal{D}_{CS}.$$

Remark 5.5. Note that the above computation of $\zeta_{\mathcal{N}}^{\mu_1}(-\infty, s)$ is justified, *a priori*, for $\mathrm{Re}s > \log_3 2$. However, by analytic continuation, it clearly follows that $\zeta_{\mathcal{N}}^{\mu_1}(-\infty, s)$ has a meromorphic continuation to all of $\mathbb{C}$ and is given by the same resulting expression for every $s \in \mathbb{C}$. Analogous comments apply to similar computations elsewhere in the paper.

Since $\partial\Omega_2$ has only one limit point, there is only one term in the corresponding multifractal zeta function at $-\infty$ for μ_2. See Figure 7. The formula is *exactly* the same as the corresponding formula for the a-String Θ_a below. This is because, in the case of both Ω_2 and Θ_a, the only closed interval of length η_n that contains infinitely many unit point-masses is $[0, \eta_n]$. So,

$$R^n(-\infty) = [0, \eta_n]$$

which means

$$J^1(-\infty) = K^1(-\infty) = [0, \eta_1]$$

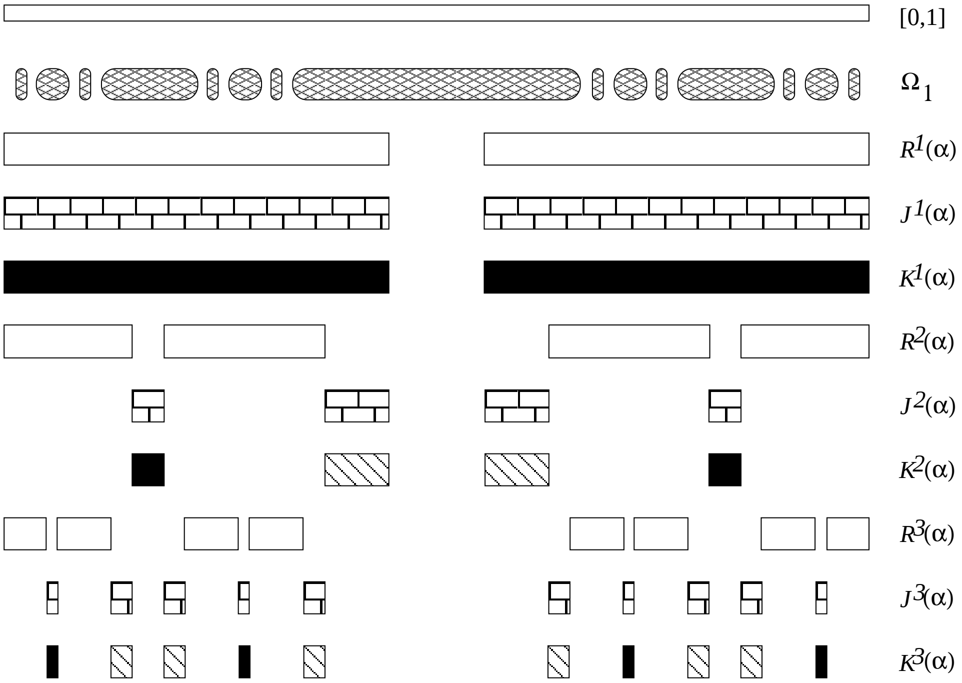

Figure 6: Construction of $\zeta_{\mathcal{N}}^{\mu_1}(-\infty, s)$.

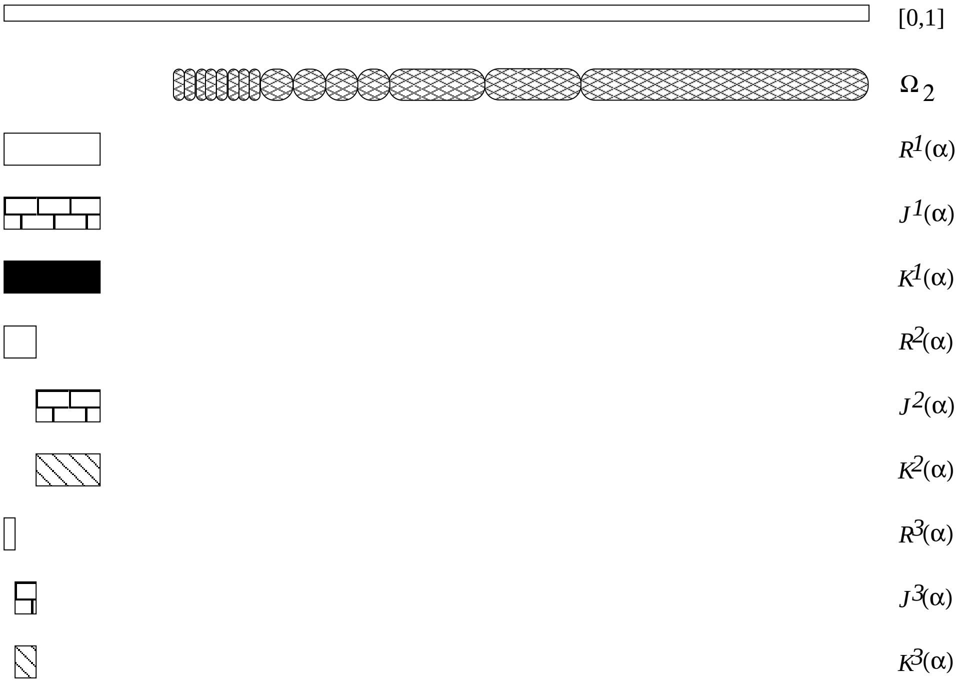

Figure 7: Construction of $\zeta_{\mathcal{N}}^{\mu_2}(-\infty, s)$.

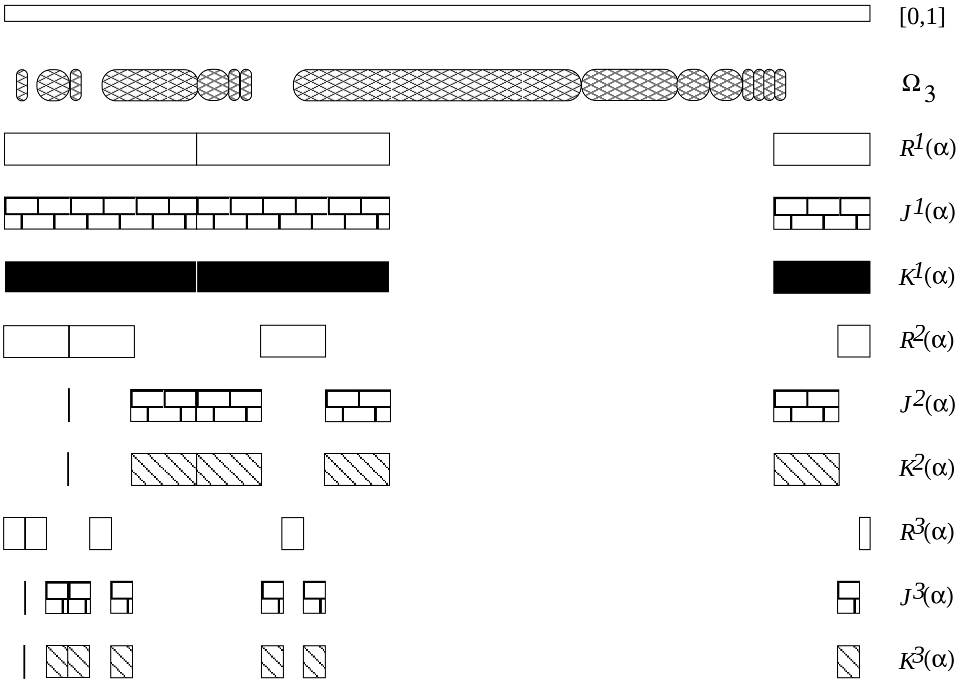

Figure 8: *Construction of* $\zeta^{\mu_3}_{\mathcal{N}}(-\infty, s)$.

and for $n \geq 2$,

$$J^n(-\infty) = (\eta_n, \eta_{n-1}].$$

All of the terms from $J^n(-\infty) = (\eta_n, \eta_{n-1}]$ are redundant. Therefore, $K^n(-\infty) = \emptyset$ for $n \geq 2$ and

$$\zeta^{\mu_2}_{\mathcal{N}}(-\infty, s) = \eta_1^s = \frac{1}{9^s}$$

which, of course, is entire and has no poles.

Theorem 5.1 does not apply to μ_3 since $\partial\Omega_3$ is not perfect, but $\partial\Omega_3$ is not as trivial as $\partial\Omega_2$. See Figure 8. The proof of the third formula in Theorem 5.1 illustrates how the regularity value $-\infty$ detects the accumulation points and the distances between accumulation points as the scales decrease to zero. For $\mathcal{N} = \{3^{-n-1}\}_{n=1}^{\infty}$, the effective lengths are zero for $n \geq 2$. This results in the following formula, in accordance with Theorem 4.8.

$$\zeta^{\mu_3}_{\mathcal{N}}(-\infty, s) \;=\; \left(\frac{1}{9}\right)^s + \left(\frac{4}{9}\right)^s.$$

The simplicity of this formula is misleading in that different choices of $\mathcal{N}$ can yield an infinite number of nonzero terms for the resulting multifractal zeta function, thus

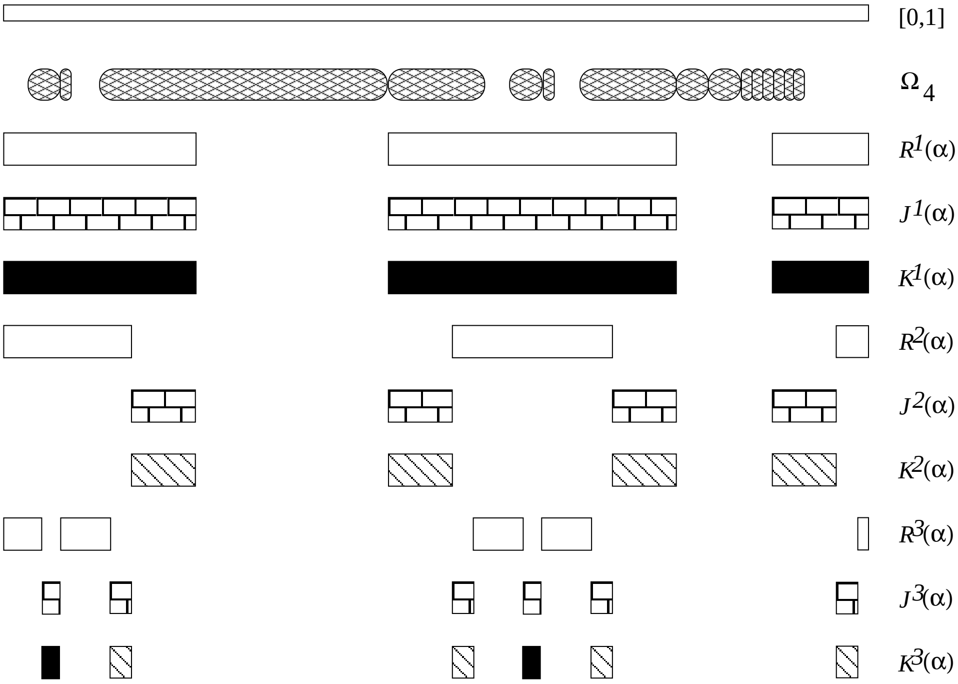

Figure 9: *Construction of* $\zeta_{\mathcal{N}}^{\mu_4}(-\infty, s)$.

it may not be entire. For instance, $\mathcal{N} = \{1/(3^{n+1}+1)\}_{n=1}^{\infty}$ satisfies the following inequalities;

$$3^{-n-2} < 1/(3^{n+1}+1) < 3^{-n-1}$$

and

$$3^{-n} - 3^{-n-1} = 2 \cdot 3^{-n-1} > 2/(3^{n+1}+1).$$

Therefore, the effective lengths $\ell_{E,k}$ would be positive for every $k \in \mathbb{N}$ and, according to Theorem 4.8, the multifractal zeta function would have infinitely many terms of the form $\ell_{E,k}^{s}$. Further, Corollary 4.10 implies that the abscissa of convergence of $\zeta_{\mathcal{N}}^{\mu_3}(-\infty, s)$ for any choice of scales $\mathcal{N}$ is bounded above by zero. Indeed, the effective string of Ω_3 is the complement of the support of the singular part of the measure σ in Section **??**, which has its abscissa of convergence at zero. The case of Ω_2 is quite different because regardless of the choice of scales $\mathcal{N}$, the multifractal zeta function consists of one term η_1^s.

For Ω_4, the multifractal zeta function is

$$\begin{aligned}
\zeta_{\mathcal{N}}^{\mu_4}(-\infty,s) &= h_4(s)+\sum_{n=2}^{\infty} m_n\left(l_{2n-1}+l_{2n}-2\eta_{2n-1}\right)^s \\
&= h_4(s)+\sum_{n=2}^{\infty} 2^{n-1}\left(\frac{1}{3^{2n-1}}+\frac{1}{3^{2n}}-\frac{2}{3^{2n}}\right)^s \\
&= h_4(s)+\sum_{n=2}^{\infty} 2^{n-1}\left(\frac{3+1-2}{3^{2n}}\right)^s \\
&= h_4(s)+\frac{2^s\cdot 2}{81^s}\sum_{n=0}^{\infty}\frac{2^n}{9^{ns}} \\
&= h_4(s)+\left(\frac{2^{s+1}}{81^s}\right)\left(\frac{1}{1-2\cdot 9^{-s}}\right),
\end{aligned}$$

where $h_4(s)$ is entire. See Figure 9. Therefore, the poles of $\zeta_{\mathcal{N}}^{\mu_4}(-\infty,s)$ are given by

$$\mathcal{D}_{\mathcal{N}}^{\mu_4}(-\infty)=\left\{\log_9 2+\frac{2i\pi m}{\log 9} \mid m\in\mathbb{Z}\right\}.$$

In this section, the sequence of scales $\mathcal{N}$ was chosen to be $\{3^{-n-1}\}_{n=1}^{\infty}$. For $q=1,2,3,4$, the multifractal zeta function of each measure μ_q with regularity $\alpha=\infty$ is equal to the geometric zeta function of the Cantor String, as follows from Theorem 3.2. Thus, the collections of poles $\mathcal{D}_{\mathcal{N}}^{\mu_q}(\infty)$ each coincide with the complex dimensions of the Cantor String.

For regularity $\alpha=-\infty$, the multifractal zeta functions (and their respective poles) differ for each $q=1,2,3,4$. Specifically, $\zeta_{\mathcal{N}}^{\mu_1}(\infty,s)$ and $\zeta_{\mathcal{N}}^{\mu_1}(-\infty,s)$ have the same collection of poles, corresponding to the fact that $\partial\Omega_1$ has equal Minkowski and Hausdorff dimensions. Both $\zeta_{\mathcal{N}}^{\mu_2}(-\infty,s)$ and $\zeta_{\mathcal{N}}^{\mu_3}(-\infty,s)$ are entire multifractal zeta functions and both $\partial\Omega_2$ and $\partial\Omega_3$ have Hausdorff dimension equal to zero. However, if a different sequence of scales were chosen, then $\zeta_{\mathcal{N}}^{\mu_3}(-\infty,s)$ could have poles, whereas, regardless of the choice of scales, $\zeta_{\mathcal{N}}^{\mu_2}(-\infty,s)$ would be entire. This reflects the fact that $\partial\Omega_3$ is topologically more complicated than $\partial\Omega_2$ in that $\partial\Omega_3$ contains countably many accumulation points and $\partial\Omega_2$ only contains one. Finally, $\zeta_{\mathcal{N}}^{\mu_4}(-\infty,s)$ has poles on a discrete line above and below the Hausdorff dimension of $\partial\Omega_4$, which is $\log_9 2$. In all of these cases, the multifractal zeta functions and their corresponding poles for regularity $-\infty$ depend on the choice of sequence of scales $\mathcal{N}$.

Example 5.6 (Regularity 0 and Variants of the Cantor String).

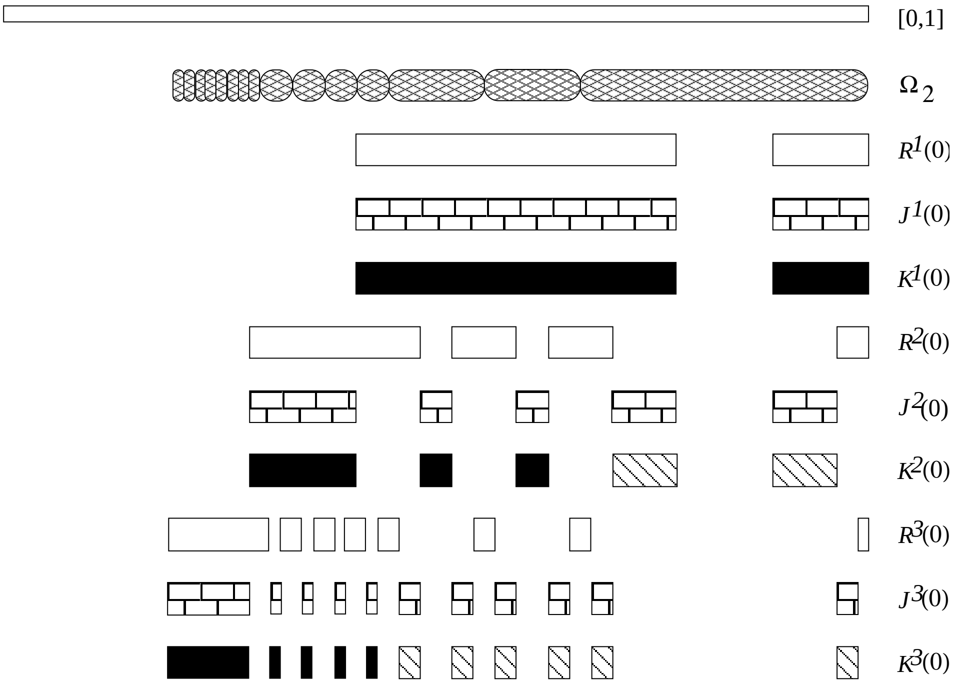

Figure 10: *Construction of* $\zeta^{\mu_2}_{\mathcal{N}}(0,s)$.

Corollary 5.2 yields a formula for the multifractal zeta function of the measure μ_1, regularity $\alpha = 0$ and sequence of scales $\mathcal{N} = \{3^{-n-1}\}_{n=1}^{\infty}$. Specifically,

$$\zeta^{\mu_1}_{\mathcal{N}}(0,s) = \sum_{n=1}^{\infty} 2(2^{n-1}3^{-(n+1)s}) = 2 \cdot 3^{-s}\zeta_{CS}(s) = 2 \cdot 3^{-s}\zeta^{\mu_1}_{\mathcal{N}}(\infty,s).$$

The coincidence between the multifractal zeta functions for the measure μ_1 and regularity values 0 and ∞ is just that; a coincidence. The form of the zeta function with regularity 0 depends heavily on the choice of scales $\mathcal{N}$ and in general can be quite complicated.

Consider the multifractal zeta function for the measure μ_2, regularity $\alpha = 0$ and sequence $\mathcal{N} = \{3^{-n-1}\}_{n=1}^{\infty}$. See Figure 10. The construction of the zeta function is complicated by the fact that every endpoint of the fractal string Ω_2 is detected by all scales that are small enough. Generally, for regularity $\alpha = 0$, if an endpoint is isolated, then intervals on both sides are picked up for all scales that are small enough to contain just the single endpoint. If an endpoint is a limit point of $\partial\Omega$, then the side which corresponds to the interior of a subinterval $(a_j, b_j) \subset \Omega$ contributes an interval of length η_n for every scale which is strictly smaller than the length of the interval ℓ_j. The multifractal zeta function for the measure μ_2, regularity $\alpha = 0$ and scales $\mathcal{N} = \{3^{-n-1}\}_{n=1}^{\infty}$ has the form

$$\begin{aligned}\zeta_{\mathcal{N}}^{\mu_2}(0,s) &= \left(\frac{10}{27}\right)^s + \left(\frac{1}{9}\right)^s \\ &\quad + \sum_{n=1}^{\infty}\left[2^n 3^{(-n-2)s} + \left((2^{n+1}-1)3^{-n-2} + 3^{-n-3}\right)^s\right].\end{aligned}$$

The multifractal zeta functions for the measures μ_3 and μ_4 with regularity $\alpha = 0$ also have very complicated structure.

Example 5.7 (**Revisiting Variants of the a-String**).

Let us revisit the examples involving the a-String. From Section 1, the a-String Θ_a for $a > 0$ is given by $\Theta_a = \bigcup_{j=1}^{\infty}((j+a)^{-a}, j^{-a})$, so $\mathcal{L}_a = \{j^{-a} - (j+1)^{-a}\}_{j=1}^{\infty}$. See Figure 3 and Example 1.11. Consider the measure

$$\mu_{\Theta_a} = \sum_{j=1}^{\infty} \delta_{j^{-a}}.$$

For any $\mathcal{N}$ such that $\eta_n \searrow 0$,

$$\zeta_{\mathcal{N}}^{\mu_{\Theta_a}}(\infty, s) = \zeta_{\mathcal{L}_a}(s)$$

and

$$\zeta_{\mathcal{N}}^{\mu_{\Theta_a}}(-\infty, s) = \eta_1^s.$$

The last equation (where $\alpha = -\infty$) follows immediately from the fact that 0 is the only limit point of the set of endpoints with unit point-mass. This same formula is derived above with a similar example, μ_2, involving the lengths of the Cantor String.

Recall from Section 1 another fractal string Θ_a^P which has the same lengths as the a-String, but which has perfect boundary. We now put a measure on $[0,1]$ that has a unit point-mass precisely at each endpoint of Θ_a^P. Let

$$\mu_{\Theta_a^P} = \sum_{j=1}^{\infty}(\delta_{a_j} + \delta_{b_j}),$$

where (a_j, b_j) are the disjoint open intervals whose union is Θ_a^P, and each endpoint has its mass counted only once. By Theorem 3.2,

$$\zeta_{\mathcal{N}}^{\mu_{\Theta_a^P}}(\infty, s) = \zeta_{\mathcal{N}}^{\mu_{\Theta_a}}(\infty, s) = \zeta_{\mathcal{L}_a}(s).$$

Since Θ_a^P has a perfect complement in $[0,1]$, the only potentially non-empty regularity values are those corresponding to closed intervals containing zero, one, two or infinitely many point-masses when $\mathcal{N}$ satisfies $\eta_n \searrow 0$, similar to the situation in Theorem 5.1. However, Theorem 5.1 does not apply since there is no sequence $\mathcal{N}$ such that $l_n > 2\eta_n$ for all $n \in \mathbb{N}$. The case of $\alpha = -\infty$ yields the following formula if $2\eta_n > l_n$ and $l_n > \eta_n > l_{n+1}$ for all $n \in \mathbb{N}$:

$$\begin{aligned}
\zeta_{\mathcal{N}}^{\mu_{\Theta_a^P}}(-\infty, s) &= 1 + (l_1 - 2\eta_2)^s \\
&\quad +(l_2 - 2\eta_3)^s \\
&\quad +(l_3 - 2\eta_4)^s \\
&\quad +(l_4 - 2\eta_5)^s \\
&\quad +(l_5 - 2\eta_7)^s \\
&\quad +...
\end{aligned}$$

The apparent lack of the term $(l_5 - 2\eta_6)^s$ in the sixth stage (fifth line above) is a result of the fact that eventually $2\eta_n > l_{n-1}$ (specifically when $n = 6$) when $\mathcal{N}$ is chosen accordingly. This means that no term involving $l_5 - 2\eta_6$ is introduced at the sixth stage. One must get to the seventh scale η_7 before a gap in the fifth open interval is introduced.

Perhaps there exists a sequence of scales $\mathcal{N}$ which generates effective lengths that are a constant multiple of the lengths of the a-string, so that $\zeta_{\mathcal{N}}^{\mu_{\Theta_a^P}}(-\infty, s)$ is an entire function times $\zeta_{\mathcal{L}_a}(s)$ plus another entire function, much as in Corollary 5.2.

For regularity $\alpha = 0$, the structure of the multifractal zeta function of the measure μ_a is very similar to that of μ_2 above because both fractal strings have boundaries with a single limit point at 0. However, the multifractal zeta function with regularity $\alpha = 0$ of the measure μ_a^P (with scales as in the example with regularity $-\infty$ above) does not mirror the structure of the corresponding multifractal zeta function from Theorem 5.1, even though $\partial\Theta_a^P$ is a perfect set. Indeed,

$$\zeta_{\mathcal{N}}^{\mu_{\Theta_a^P}}(0, s) = \sum_{j=1}^{\infty} \left[\ell_j^s + 2(\ell_j - \eta_n(j))^s\right],$$

where $\eta_n(j) = \max_{n\in\mathbb{N}}\{\eta_n \mid \eta_n < \ell_j\}$. This structure comes from the fact that multifractal zeta functions detect intervals as they fill-in (ℓ_j) and split $(2(\ell_j - \eta_n(j))$.

This section further illustrates the dependence of the multifractal zeta functions for a measure μ_Ω with regularity $\alpha = -\infty$ and 0 on the topological configuration of the fractal string Ω in question, as well as the choice of scales $\mathcal{N}$ used to examine the fractal string.

6 Examples of Simple Mass Distributions

Recall the mass distribution with support on the boundary of the a-String Θ_a (with $a = 1$) which was considered in Example **??**,

$$\nu = \lambda + \sum_{j=1}^{\infty} c_j \delta_{j^{-1}}.$$

According to Theorem 5.2 of [20],

$$\zeta_{\mathcal{N}}^{\nu}(1, s) = \zeta_{\Theta_a}(s).$$

To put the measure ν into a context more appropriate for this work, consider its singular part ν_1 (with respect to the Lebesgue measure),

$$\nu_1 = \sum_{j=1}^{\infty} c_j \delta_{j^{-1}}.$$

Thus, according to Theorem 3.2 of this text,

$$\zeta_{\mathcal{N}}^{\nu_1}(\infty, s) = \zeta_{\mathcal{N}}^{\nu}(1, s) = \zeta_{\mathcal{L}_a}(s).$$

What about other regularity values? For $\alpha = -\infty$, note that there are no closed intervals which contain infinite mass. Thus, regardless of the choice of $\mathcal{N}$,

$$\zeta_{\mathcal{N}}^{\nu_1}(-\infty, s) = \zeta_{\mathcal{N}}^{\nu}(-\infty, s) = 0.$$

For other regularity values, the structure of the multifractal zeta functions depend heavily on the weights $\{c_j\}_{j=1}^{\infty}$ and the structure of the support of the measure. In

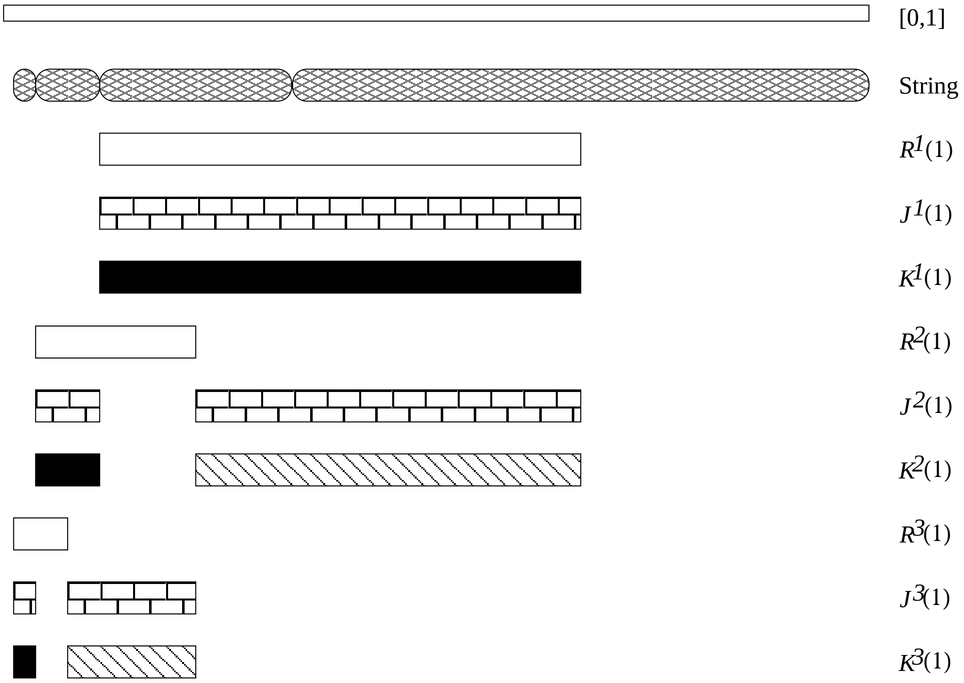

Figure 11: *Construction of* $\zeta_{\mathcal{N}}^{\sigma_1}(1,s)$.

order to clarify this dependence, recall the measure σ in Example **??**,

$$\sigma = \lambda + \sum_{n=1}^{\infty} 3^{-n} \delta_{3^{-n}}.$$

As above, consider just the singular part of this measure, denoted σ_1. The complement of the support of σ_1 is the fractal string Ω_σ with endpoints at $0, 1$ and 3^{-n} for every $n \in \mathbb{N}$. There is a point-mass at each endpoint of Ω_σ with mass equal to the coordinate of that endpoint. That is, consider the measure

$$\sigma_1 = \sum_{n=1}^{\infty} 3^{-n} \delta_{3^{-n}}.$$

This measure has a nontrivial (not entire) multifractal zeta function for regularity $\alpha = 1$ and scales $\mathcal{N} = \{3^{-n}\}_{n=1}^{\infty}$.

In this situation, each closed interval $U \in \mathcal{R}_{3^{-n}}(1)$ that contains just the endpoint/point-mass 3^{-n} has regularity 1. The structure of each $R^n(1)$ is fairly simple:

$$R^n(1) = (3^{-n-1}, 2 \cdot 3^{-n}].$$

At each new scale, we pick up a length equal to the distance between the endpoints at the value equal to that scale and the one after it. This means, for $n \geq 2$, $K^n(1)$ is one interval of length $3^{-n} - 3^{-n-1} = 2 \cdot 3^{-n-1}$. See Figure 11. Thus,

$$\begin{aligned} \zeta_{\mathcal{N}}^{\sigma_1}(1,s) &= \left(\frac{1}{3}+\frac{2}{9}\right)^s + \sum_{n=2}^{\infty}\left(\frac{2}{3^{n+1}}\right)^s \\ &= \left(\frac{5}{9}\right)^s + \left(\frac{2}{27}\right)^s \sum_{n=0}^{\infty} 3^{-ns} \\ &= \left(\frac{5}{9}\right)^s + \left(\frac{2}{27}\right)^s \left(\frac{1}{1-3^{-s}}\right). \end{aligned}$$

The poles of $\zeta_{\mathcal{N}}^{\sigma_1}(1,s)$ are

$$\mathcal{D}_{\mathcal{N}}^{\sigma_1}(1,\mathbb{C}) = \left\{\frac{2\pi i m}{\log 3} \mid m \in \mathbb{Z}\right\}.$$

Note that the support of σ_1 is $\{0,1\} \cup \{3^{-n}\}_{n\in\mathbb{N}}$. Thus σ_1 carries no weight on the fractal string $\Omega_\sigma = (supp(\sigma_1))^c$. In this case, as mentioned in Section **??**, the closed intervals that carry no weight are said to have regularity ∞. By Theorem 3.2, we have $\zeta_{\mathcal{N}}^{\sigma_1}(\infty,s) = \zeta_{\mathcal{L}_\sigma}(s)$ for any sequence $\mathcal{N}$ such that decreases to zero. The sequence $\mathcal{L}_\sigma$ is $\{2 \cdot 3^{-n}\}_{n\in\mathbb{N}}$ where each length has multiplicity one. Therefore

$$\begin{aligned} \zeta_{\mathcal{N}}^{\sigma_1}(\infty,s) &= \sum_{n=1}^{\infty}(2\cdot 3^{-n})^s \\ &= 2^s \sum_{n=1}^{\infty} 3^{-ns} \\ &= \frac{2^s \cdot 3^{-s}}{1-3^{-s}}. \end{aligned}$$

The poles of $\zeta_{\mathcal{N}}^{\sigma_1}(\infty,s)$ are the same as $\zeta_{\mathcal{N}}^{\sigma_1}(1,s)$. Specifically,

$$\mathcal{D}_{\mathcal{N}}^{\sigma_1}(1,\mathbb{C}) = \mathcal{D}_{\mathcal{N}}^{\sigma_1}(\infty,\mathbb{C}) = \left\{\frac{2\pi i m}{\log 3} \mid m \in \mathbb{Z}\right\}.$$

If the point-masses are $c^n 3^{-n}$ where $c > 0$, the measure is then

$$\sigma_c = \sum_{n=1}^{\infty} c^n 3^{-n} \delta_{3^{-n}}.$$

The regularity value

$$\alpha_c = \frac{\log\left(c^n 3^{-n}\right)}{\log\left(3^{-n}\right)} = 1 - \frac{\log c}{\log 3}$$

and the same sequence scales $\mathcal{N}$ yield the exact same nontrivial multifractal zeta function as above for this new measure σ_c. Thus,

$$\zeta_{\mathcal{N}}^{\sigma_c}(\alpha_c, s) = \left(\frac{5}{9}\right)^s + \left(\frac{2}{27}\right)^s \left(\frac{1}{1-3^{-s}}\right).$$

7 Infinitely Many Multifractal Zeta Functions

This section discusses the family of multifractal zeta functions for a certain measure γ and a specific choice of scales $\mathcal{N}$. This family contains a countably infinite number of multifractal zeta functions with poles, parameterized by the collection of regularity values $\{\alpha_j\}_{j=1}^{\infty}$ defined below. For this section, fix $\mathcal{N} = \{3^{-n}\}_{n=1}^{\infty}$.

The complement of the support of γ, the fractal string Ω_γ, has the same sequence of lengths as the Cantor String. Further, Ω_γ is a countable union of scaled copies of the complement of the measure σ_1 from Section 6. Accordingly, the boundary $\partial\Omega_\gamma$ contains infinitely many limit points. To create each of the limit points in this case, the lengths are arranged adjacent to each other in *strictly decreasing* consecutive order from left to right. That is, there is one sequence of consecutive open intervals that begins with an interval of length 1/3, one that begins with length 1/9, and two that begin with lengths of 1/27. In general there are 2^{k-1} sequences of open intervals that begin with an interval of length $1/3^{k+1}$ for each $k \in \mathbb{N}$. The set of limit points of $\partial\Omega_\gamma$ is $\{0, 1, 1/2, 1/3, 5/18, 2/9...\}$. See Figure 12.

Before constructing the measure γ, consider a collection of masses $\{c_j\}_{j=1}^{\infty}$ that satisfies a condition similar to the notion of *distinct regularity* from [20]. These masses relate directly to the countable collection of regularity values $\{\alpha_j\}_{j=1}^{\infty}$ where

$$\alpha_j = \frac{\log c_j^n}{\log 3^{-n}} = -\log_3 c_j,$$

for every $n \in \mathbb{N}$. Each α_j yields a multifractal zeta function with poles for the measure γ and scales $\mathcal{N}$. To guarantee that the structure of each of the multifractal zeta functions is as described below, each mass c_j must differ from any other mass in such a way that a given regularity α_j occurs only for closed intervals of length 3^{-n} that contain precisely the endpoint with mass c_j^n and no other mass.

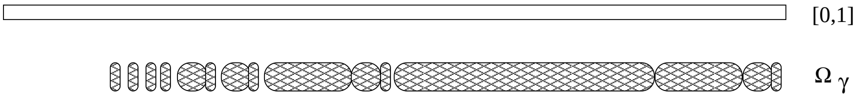

Figure 12: *Construction of the fractal string* Ω_γ.

Generically, with a measure that is supported on $\partial\Omega_\gamma$, a random collection of masses $\{c_j\}_{j=1}^\infty$ will generate such a correlation (or perhaps, a lack thereof), as can be seen using a cardinality argument and the following fact: For a fixed scale η_n, any closed interval with such length can contain either no mass, mass which is a total of finitely many consecutive powers of certain c_j, or infinitely many masses. If the interval contains an infinite number of some c_j^n for a fixed j, then it contains the whole tail for that collection, perhaps less some finite number of the first terms. Thus, each scale belongs to a countable collection of $\mathcal{R}_{\eta_n}(\alpha)$ where α ranges over all extended real numbers. Since there only a countable number of scales under consideration, a given regularity will, generically, occur only as prescribed above.

To construct γ, consider each of the sequences of consecutive open intervals that comprise Ω_γ one at a time from right to left. At the right endpoint of the interval with length 1/3, put a mass of size c_1. For each endpoint which lies to the right from there, put a mass of size c_1^n consecutively. For the next sequence of open intervals (which has first and largest length 1/9), put the mass c_2 on the right endpoint of the interval with length 1/9 and mass c_2^n on each endpoint to the right, consecutively. Continuing this process for each sequence of consecutive open intervals and mass c_j yields the measure γ. The total mass of γ is the double sum $\sum_{j=1}^\infty \sum_{n=1}^\infty c_j^n$.

To construct the family of multifractal zeta functions parameterized by the regularity values $\{\alpha_j\}_{j=1}^\infty$ for the measure γ and sequence $\mathcal{N}$, recall the measure σ_1 from Section 6. The construction of each multifractal zeta function $\zeta_\mathcal{N}^\gamma(\alpha_j, s)$ (for every $j \in \mathbb{N}$) is nearly identical to that of $\zeta_\mathcal{N}^{\sigma_1}(1, s)$. Indeed, the fractal string Ω_γ is a union of a countable collection of scaled copies of Ω_σ, which is the complement of the support of the measure σ_1.

Consider the regularity value $\alpha_1 = -\log_3 c_1$ and the multifractal zeta function $\zeta_\mathcal{N}^\gamma(\alpha_1, s)$. See Figure 13. The sets $R^1(\alpha_1)$ and $K^1(\alpha_1)$ consist of the union all closed intervals of length 1/3 which contain only the endpoint with mass c_1 (which is the point $x = 5/6$). Thus, $R^1(\alpha_1)$ has one component of length $1/3+1/9 = 4/9$. Likewise, each $R^n(\alpha_1)$ has one component of length $3^{-n} + 3^{-n-1} = 4 \cdot 3^{-n-1}$. After taking the symmetric differences and removing the redundant terms, each $K^n(\alpha_1)$ with $n > 1$

consists of one component of length 3^{-n-2}. Therefore,

$$\begin{aligned}\zeta_{\mathcal{N}}^{\gamma}(\alpha_1, s) &= \left(\frac{4}{9}\right)^s + \sum_{n=1}^{\infty}\left(\frac{1}{3^{n+2}}\right)^s \\ &= \left(\frac{4}{9}\right)^s + \left(\frac{1}{27}\right)^s \sum_{n=1}^{\infty}\left(\frac{1}{3^s}\right)^{n-1} \\ &= 4^s \cdot 9^{-s} + \frac{27^{-s}}{1-3^{-s}}.\end{aligned}$$

Every other multifractal zeta function in this parameterized family is constructed in a similar fashion. For each $j > 1$, the first nonempty $R^n(\alpha_j)$ occurs for $n = k$, where k and j are nonnegative integers that satisfy

$$2^{k+1} \geq j > 2^k.$$

Thus, $R^k(\alpha_j) = K^k(\alpha_j)$ is a single interval of length $3^{-k} + 3^{-k-1} = 4 \cdot 3^{-k-1}$. As a result, each $K^n(\alpha_j)$ with $n > k$ consists of one component of length 3^{-n-1}. The multifractal zeta function of the measure γ, sequence $\mathcal{N}$ and regularity value α_j is

$$\zeta_{\mathcal{N}}^{\gamma}(\alpha_j, s) = 4^s \cdot 3^{-(k+2)s} + \frac{3^{-(k+2)s}}{1-3^{-s}},$$

where $2^{k+1} \geq j > 2^k$.

The coincidence of the forms of these multifractal zeta functions yields an interesting relationship between the collection of their poles. Note, for all $n > j_1 > j_2 \in \mathbb{N}$,

$$|K^n(\alpha_{j_1})| = |K^n(\alpha_{j_2})| = |K_1^n(\alpha_{j_1})| = |K_1^n(\alpha_{j_2})| = 3^{-n}.$$

Therefore, ultimately, their multifractal zeta functions have exactly the same asymptotic behavior. The complex dimensions with parameters α_j all coincide. Namely, for all $j \in \mathbb{N}$,

$$\mathcal{D}_{\mathcal{N}}^{\gamma}(\alpha_j) = \left\{\frac{2m\pi i}{\log 3}\right\}_{m\in\mathbb{Z}.}$$

This behavior is a result of the fact that the fractal string Ω_γ is a union of a countable collection of scaled copies of the same fractal string.

Example 7.1 (The Fractal String Ω_γ).

Regularity $\alpha = \infty$ yields the geometric zeta function of the fractal string Ω_γ, which has the same sequence of lengths as the Cantor String. Therefore,

$$\zeta_{\mathcal{N}}^{\gamma}(\infty, s) = \zeta_{CS}(s) = \frac{3^{-s}}{1-2\cdot 3^{-s}}.$$

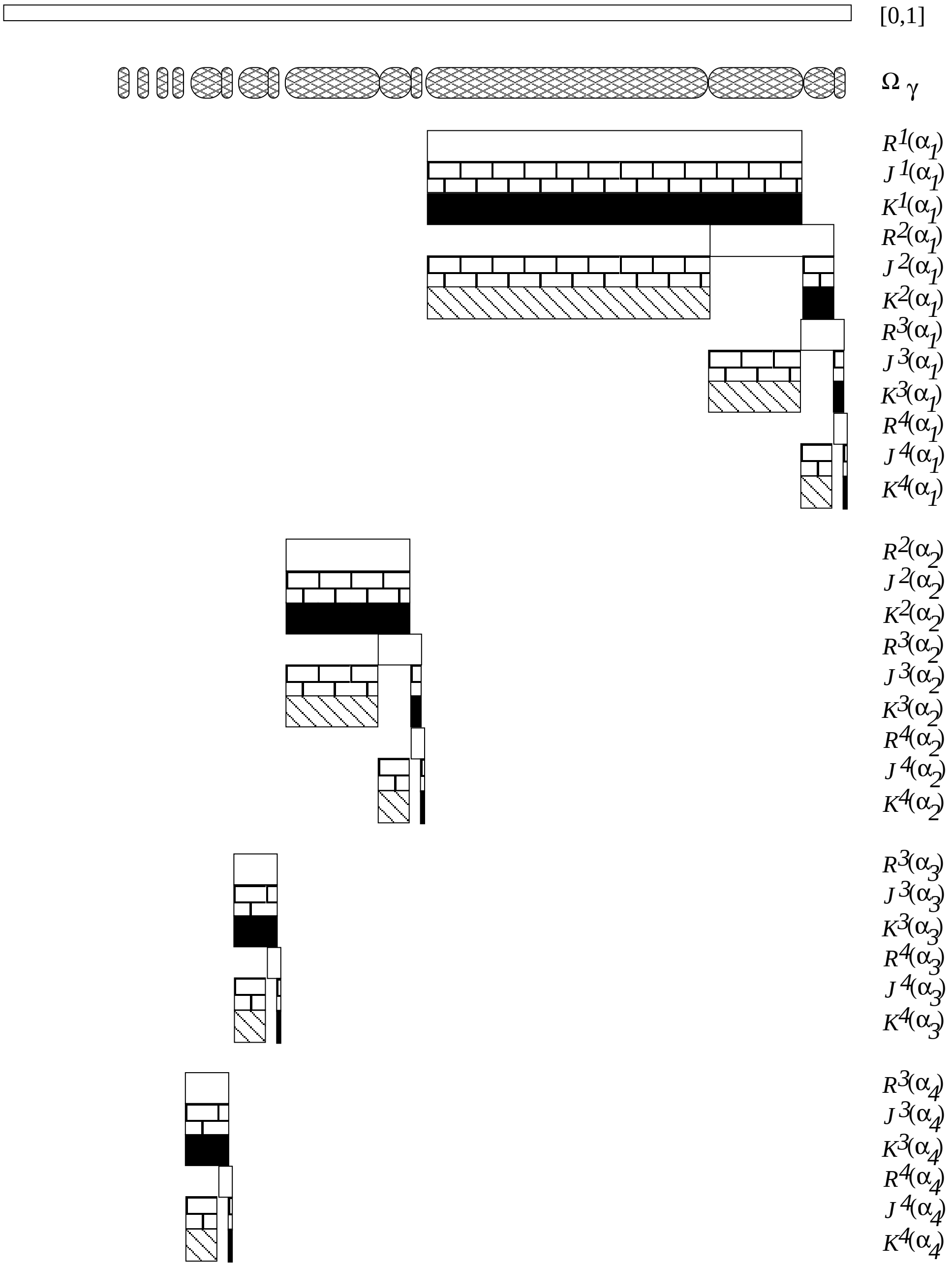

Figure 13: *Construction of multifractal zeta functions for the measure* γ.

The (usual) complex dimensions dimensions of the Cantor String are the complex dimensions with parameter ∞ of the measure γ and sequence $\mathcal{N}$,

$$\mathcal{D}_{\mathcal{N}}^{\gamma}(\infty) = \mathcal{D}_{CS} = \left\{ \log_3 2 + \frac{2m\pi i}{\log 3} \right\}_{m \in \mathbb{Z}.}$$

The support of the measure γ has more interesting structure. Consider the measure μ_γ where

$$\mu_\gamma = \mu_{\Omega_\gamma}$$

This measure has a unit-point mass at every endpoint of the fractal string Ω_γ, which is the complement of the support of the measure γ. This fractal string is an example where the result of Corollary 4.5 does not hold. The Hausdorff dimension of $\partial\Omega_\gamma$ is zero since it is a countable set, yet the structure of the effective string $\Omega_{\gamma,E}$ is complicated enough to have a positive Minkowski dimension. We have

$$0 = \dim_H(\partial\Omega_\gamma) < \dim_M(\partial\Omega_{\gamma,E}) = \log_3 2.$$

The lengths of this effective string are

$$\mathcal{L}_{\gamma,E} = \{1/2, 1/6, 1/18, 1/18, 1/54, 1/54, 1/54, 1/54, ...\}.$$

That is, there is one interval with length $1/2$, and 2^{n-1} intervals with length $2^{-1}3^{-n}$ for every $n \in \mathbb{N}$. Therefore, the geometric zeta function for the effective string $\Omega_{\gamma,E}$ is

$$\begin{aligned} \zeta_{\mathcal{L}_{\gamma,E}}(s) &= \left(\frac{1}{2}\right)^s + \frac{1}{2\cdot 3^s}\sum_{n=0}^{\infty}\left(\frac{2}{3^s}\right)^n \\ &= \left(\frac{1}{2}\right)^s + \left(\frac{1}{2\cdot 3^s}\right)\frac{1}{1-2\cdot 3^{-s}}. \end{aligned}$$

The abscissa of convergence of this geometric zeta function yields the desired Minkowski dimension.

Remark 7.2. The result of Corollary 4.5 is not entirely lost on this example. Consider the effective string of the effective string, denoted $\Omega_{\gamma,E,E}$. The only limit point in the complement of this fractal string is 0, which has Minkowski dimension 0 (the geometric zeta function of $\Omega_{\gamma,E,E}$ consists of a single term of the form η_1^s). Thus,

$$\dim_H(\partial\Omega_\gamma) = \dim_M(\partial\Omega_{\gamma,E,E}) = 0.$$

For strings with perfect boundary, the effective string is identical to the original string. A deeper relationship between effective strings and the Hausdorff dimension than Corollary 4.5 must certainly exist.

This section verifies that the multifractal zeta functions can have interesting structure for a fixed measure and fixed choice of scales, parameterized by an infinite collection of regularity values, even with the relatively simple structure of the measure γ being comprised entirely of point-masses. The next section investigates the multifractal zeta functions for measures that are quintessential examples of multifractals.

8 Mass Distributions and Partition Zeta Functions

Multifractal zeta functions can be used to analyze measures which are truly multifractal, such as the measure μ discussed in this section (and found on page 255 in [8]). The multifractal zeta functions for these measures detect fractal structure for an infinite number of regularity values, but they do not appear to reflect the obvious symmetries in the construction of the support of the measures and their distribution of mass. This leads to the definition of a new family of zeta functions called *partition zeta functions*. The construction of the partition zeta functions also depends on regularity values, but attention is restricted to the intervals in a family of partitions of the unit interval. Before defining this new type of zeta function, consider the following examples of measures which are supported on the Cantor Set $\partial\Omega_1$.

Example 8.1 (Mass Distributions on the Cantor Set).

The Cantor Set is the intersection of a decreasing sequence of closed sets $\{E_q\}_{q=0}^{\infty}$. See Figure 14 and the construction in [8]. The construction lends itself readily to the definition of a mass distribution on the resulting fractal set. First, consider a mass distribution ρ on the Cantor Set which is constructed by assigning mass 2^{-q} to each subinterval E_q^r of the closed set E_q. These subintervals have length 3^{-q}, with multiplicity 2^q. Thus, it is natural to consider the regularity value α that corresponds to this combination of mass and length:

$$\alpha = \frac{\log 2^{-q}}{\log 3^{-q}} = \frac{\log 2}{\log 3} = \log_3 2,$$

for every $q \in \mathbb{N}$. That is, every closed subinterval E_q^r at every stage q has the same regularity $\log_3 2$.

Figure 14: *Construction of the Cantor Set.*

Remark 8.2. The measure ρ is the natural Hausdorff measure of the Cantor Set. The primitive of ρ defined by $N_\rho(x) = \int_0^x (d\rho) = \rho([0,x])$ is a function whose graph is the well-known Devil's Staircase. This function is continuous on $[0,1]$ with zero derivative almost everywhere (on the Cantor String, in fact), yet its range is the full interval $[0,1]$. See §12.2 in [29] for an investigation of the Devil's Staircase and a discussion of a new notion of fractality.

Two interesting multifractal zeta functions for the mass distribution ρ and sequence of scales $\mathcal{N} = \{3^{-n}\}_{n=1}^{\infty}$ are the ones with regularity values ∞ and $\log_3 2$:

$$\begin{aligned}\zeta_{\mathcal{N}}^{\rho}(\infty, s) &= \zeta_{CS}(s), \text{ and}\\ \zeta_{\mathcal{N}}^{\rho}(\log_3 2, s) &= \frac{2}{3^s} + \sum_{n=2}^{\infty} \frac{2^{n-1}}{3^{ns}}\\ &= \frac{2}{3^s} + \frac{2 \cdot 3^{-2s}}{1 - 2 \cdot 3^{-s}}.\end{aligned}$$

The first equation follows from Theorem 3.2 and the second follows from the fact that the closed subintervals E_q^r are the only intervals of length 3^{-q} that have exactly mass 2^{-q}. All other closed intervals with length 3^{-q} have too little mass. The simplicity of this structure is not found in the following measure μ.

There is an infinite collection of multifractal zeta functions which may have poles for the measure μ which is a multifractal mass distribution supported on the Cantor Set. See Figure 15 for the first few steps in the construction of this measure. Theorem 3.2 yields

$$\zeta_{\mathcal{N}}^{\mu}(\infty, s) = \zeta_{CS}(s),$$

for any sequence of scales $\mathcal{N}$ that tends to zero. That is, regularity ∞ yields the geometric zeta function of the complement of the support of the measure μ.

The measure μ is a result of putting mass on each closed interval from the construction of the Cantor Set, with 1/3 of the total mass from the previous stage going

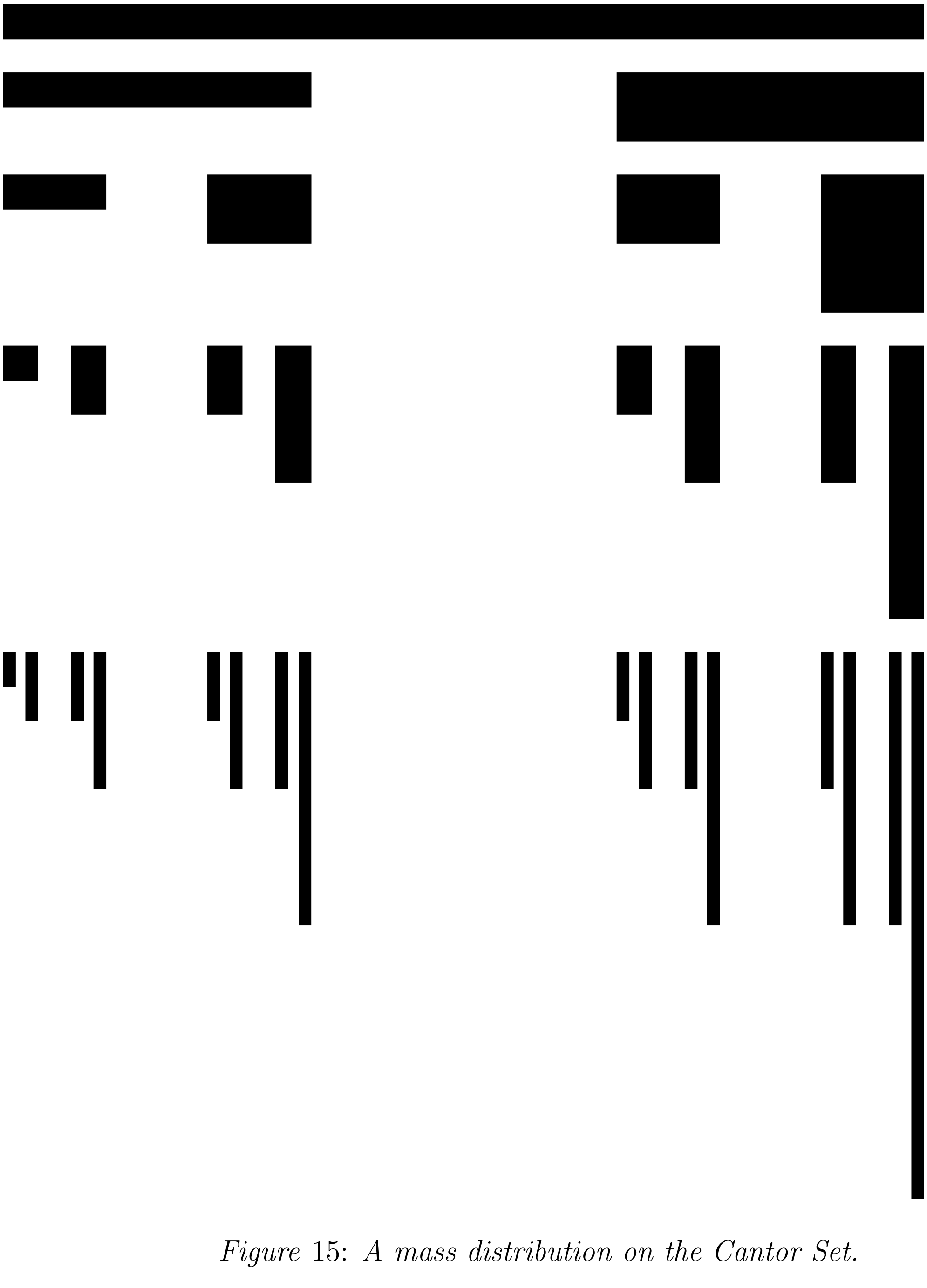

Figure 15: *A mass distribution on the Cantor Set.*

to the left subinterval and 2/3 to the right. At every stage $q \in \mathbb{N}$ there are 2^q closed intervals of length 3^{-q}, which are the E_q^r. Of these,

$$\binom{q}{p} = \frac{q!}{p!(q-p)!}$$

have weight $(1/3)^{q-p}(2/3)^p = 2^p/3^q$ where $p \in \{0, ..., q\}$.

Every interval E_q^r has regularity which corresponds to its length 3^{-q} and its mass $2^{p_r}/3^q$, where $p_r \in \{0, ..., q\}$ is the unique exponent corresponding to the mass of E_q^r. To determine the intervals E_q^r which have the same regularity, consider the fact that

$$\alpha = \alpha(k_1, k_2) = \frac{\log(2^{nk_1}/3^{nk_2})}{\log(1/3^{nk_2})} = 1 - \frac{k_1}{k_2}\log_3 2,$$

for all $n \in \mathbb{N}$ and where $k_1 \in \mathbb{N} \cup \{0\}$, $k_2 \in \mathbb{N}$ and $k_1 < k_2$. These regularity values $\alpha(k_1, k_2)$ yield multifractal zeta functions which may have poles for the measure μ and scales $\mathcal{N}$ when $k_1 \geq 1$ and $k_2 \geq 2$. Note that these regularity values depend on the reduced form of the fraction k_1/k_2. Indeed, given a pair (k_1', k_2') that is not relatively prime and a pair (k_1, k_2) that is relatively prime which together satisfy

$$\frac{k_1'}{k_2'} = \frac{k_1}{k_2},$$

it is necessary to consider the relatively prime pair (k_1, k_2) to construct the multifractal zeta function (and the partition zeta function below) appropriately. For further details, see Remark 8.3.

For every interval E_q^r there is a unique triple $(n, k_1, k_2) \in \mathbb{N}^3$ where k_1 and k_2 are relatively prime and the following hold:

$$A(E_q^r) = \frac{\log(2^p/3^q)}{\log(1/3^q)} = \frac{\log(2^{nk_1}/3^{nk_2})}{\log(1/3^{nk_2})} = \alpha(k_1, k_2),$$

where $p = p_r$. That is, each interval E_q^r is a subset of $R^q(\alpha(k_1, k_2))$ where $p = p_r = nk_1$ and $q = nk_2$.

Remark 8.3. Consider the binomial coefficient $\binom{q}{p}$ where $p \in \{0, ..., q\}$ and $q \in \mathbb{N}$. The number of intervals in the set E_q which have the same regularity corresponding to the exponent p is given by $\binom{q}{p}$. According to the breakdown above, in order to guarantee that every interval from every stage for a fixed regularity $\alpha(k_1, k_2)$ is collected, the pair (k_1, k_2) must be relatively prime, otherwise terms which correspond to some sets with the correct regularity will be left out. For instance, suppose the pair (k_1', k_2') is not relatively prime and

$$\frac{k_1'}{k_2'} = \frac{k_1}{k_2}$$

where the pair (k_1, k_2) is relatively prime. Then $\alpha(k_1', k_2') = \alpha(k_1, k_2)$, and there exists a unique $n' \in \mathbb{N}$ such that $k_1' = n'k_1$ and $k_2' = n'k_2$. There also exists some integer m such that $mk_2 \neq nn'k_2$ for every n. That is, the pair (k_1, k_2) determined by the regularity must be relatively prime otherwise the corresponding terms would not represent all of the intervals with the prescribed regularity.

For the sake of exposition, consider the case where $k_1 = 1$ and $k_2 = 2$. For all $n \in \mathbb{N}$, the intervals in the construction of the measure μ with length $1/3^{2n}$ and weight $2^n/3^{2n}$ have regularity $\alpha(1, 2)$ (see Figure 16). However, *these are not the only intervals with such regularity.* Indeed, there is an interval of some given length $\eta_q = 3^{-q}$ for every $q \in \mathbb{N}$ which does not correspond to any interval E_q^r from the construction of the Cantor Set, yet has regularity $\alpha(1, 2)$. This fact can be seen by considering the primitive of the measure μ, which is a continuous function. Therefore, shifting one of closed intervals E_q^r slightly left or right continuously changes the regularity up to ∞ (corresponding to a lack of mass). This leads to additional structure which is difficult to completely describe and does not reflect the symmetry of the construction of μ. For instance, note that the number of intervals E_q^r of length $\eta_q = 3^{-q}$ which have certain mass is given by a binomial coefficient. Then,

$$\binom{nk_2}{nk_1} = \binom{nk_2}{n(k_2 - k_1)} = \frac{(nk_2)!}{(nk_1)!(n(k_2 - k_1))!},$$

for every $n, k_1, k_2 \in \mathbb{N}$ where k_1 and k_2 are relatively prime. This symmetry is not seen in the family of multifractal zeta functions parameterized by the $\alpha(k_1, k_2)$ due to the terms that arise from the closed intervals which do not correspond to any of the E_q^r.

To deal with this excess structure in the construction of the multifractal zeta functions, restrict the allowable intervals to a family of partitions, such as one which is determined by the construction of the measure. Consider an ordered family of partitions $\mathfrak{P} = \{\mathcal{P}_n\}_{n=1}^{\infty}$ which split the unit interval into finitely many subintervals. The order is given by the relation $\mathcal{P}_n \succ \mathcal{P}_{n+1}$ taken to mean that the each of the intervals P_{n+1}^k which comprise the partition $\mathcal{P}_{n+1}$ is a subinterval of some interval in $\mathcal{P}_n$. The lengths of the intervals in such a family of partitions can be used to generate a new zeta function.

Definition 8.4. *For a measure μ on the interval $[0, 1]$ with an ordered family of partitions $\mathfrak{P}$, the partition zeta function with regularity α is*

$$\zeta_{\mathfrak{P}}^{\mu}(\alpha, s) = \sum_{n=1}^{\infty} \sum_{k=1}^{p_n} \delta_\alpha(P_n^k)|P_n^k|^s,$$

where $\delta_\alpha(P)$ equals 1 if $A(P) = \alpha$ and equals 0 otherwise, p_n is the number of intervals in the partition $\mathcal{P}_n$, and Re*s is large enough.*

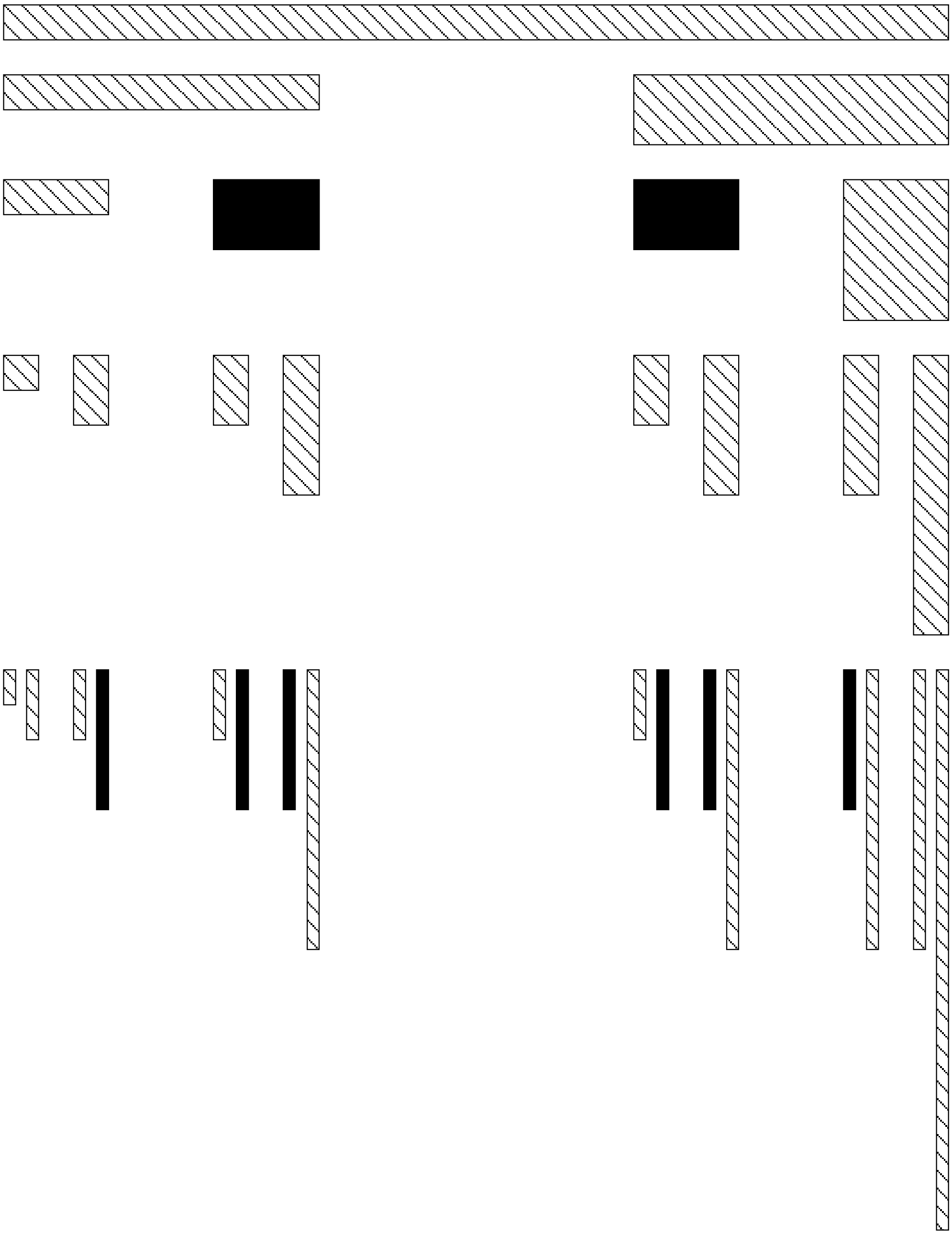

Figure 16: *Intervals* E_q^r *with regularity* $\alpha(1,2)$.

Remark 8.5. There is not a particular partition zeta function that directly corresponds to the lengths of some fractal string associated with a given measure, as is the case in Theorem 3.2 and multifractal zeta functions. There is a connection, however, as seen in Theorem 8.9. Also, a given family of partitions $\mathfrak{P}$ will determine at most countably many regularity values and partition zeta functions (which are not identically zero) for a given measure.

Collecting the lengths according to certain regularity values and the partition family $\mathfrak{P}$ maintains the symmetry in the construction of the measure μ when $\mathfrak{P}$ is the collection of triadic intervals. For definiteness, the individual partitions in this family are taken to be the closed intervals E_q^r and their complements in the unit interval. For the uniform mass distribution ρ and this $\mathfrak{P}$, the partition zeta function with regularity value $\alpha = \log_3 2$ is

$$\zeta_{\mathfrak{P}}^{\rho}(\log_3 2, s) = \sum_{n=1}^{\infty} \left(\frac{2^n}{3^{ns}} \right) = \frac{2 \cdot 3^{-s}}{1 - 2 \cdot 3^{-s}}.$$

The partition zeta functions of the measures μ and ρ with regularity ∞ is given by a collection of intervals which has a sum of lengths that rapidly diverges. That is, since there is no symmetric difference being taken (as in the case of the multifractal zeta functions), the full middle third length with no mass appears in each partition $\mathcal{P}_n$, thus the zeta function has the form

$$\zeta_{\mathfrak{P}}^{\rho}(\infty, s) = \zeta_{\mathfrak{P}}^{\mu}(\infty, s) = \frac{1}{3^s} + \frac{5}{9^s} + \frac{19}{27^s} \cdots$$

which diverges for $s = 1$.

For the measure μ, the partition zeta functions have nice structure both individually and collectively. Consider again the regularity value $\alpha(1,2)$ and the family $\mathfrak{P}$ of partitions given by the triadic intervals. See Figure 16. This regularity value occurs only in the partitions $\mathcal{P}_{2n}$, with the total number of intervals in each partition with regularity $\alpha(1,2)$ given by the binomial coefficient $\binom{2n}{n}$. Therefore,

$$\zeta_{\mathfrak{P}}^{\mu}(\alpha(1,2), s) = \sum_{n=1}^{\infty} \binom{2n}{n} 3^{-2ns}.$$

The form of the partition zeta function above where $k_1 = 1$ and $k_2 = 2$ generalizes to $k_1 \geq 0$, $k_2 \geq 1$ and $k_1 < k_2$. The regularity value $\alpha(k_1, k_2)$ with fixed (k_1, k_2) which are relatively prime only occurs in the partitions $\mathcal{P}_{nk_2}$ for every each $n \in \mathbb{N}$, with multiplicity $\binom{nk_2}{nk_1}$. In summation,

$$\zeta_{\mathfrak{P}}^{\mu}(\alpha(k_1, k_2), s) = \zeta_{\mathfrak{P}}^{\mu}(\alpha(k_2 - k_1, k_2), s) = \sum_{n=1}^{\infty} \binom{nk_2}{nk_1} 3^{-k_2 ns}.$$

For the extreme cases when $k_1 = 0$ and $k_2 = 1$ or $k_1 = k_2 = 1$, the partition zeta functions pick up one length from each partition in $\mathfrak{P}$. Therefore,

$$\zeta^{\mu}_{\mathfrak{P}}(1, s) = \zeta^{\mu}_{\mathfrak{P}}(1 - \log_3 2, s) = \sum_{n=1}^{\infty} 3^{-ns} = \frac{3^{-s}}{1 - 3^{-s}}.$$

Hence, the abscissa of convergence is zero.

It remains to determine the abscissae of convergence for partition zeta functions with other regularity values. In the case of the measures μ and ρ, the only intervals in the family of triadic intervals $\mathfrak{P}$ which contribute to regularity values which are not ∞ have total length $\sum_{n=1}^{\infty} 2^n \cdot 3^{-n} = 2$. This means that each partition zeta function of the measures μ and ρ with regularity other than ∞ and the collection of triadic intervals converges for all s such that $\mathrm{Re} s \geq 1$.

A combination of Stirling's formula and the root test determine a formula for the abscissae of convergence. Recall Stirling's formula, which states that

$$\lim_{n\to\infty} \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} = 1.$$

Thus, before applying the n-th root test, the general n-th term of the partition zeta function of the measure μ with regularity $\alpha(k_1, k_2)$ and the triadic intervals as the partition family $\mathfrak{P}$ can be approximated as follows:

$$\begin{aligned}\binom{nk_2}{nk_1} 3^{-nk_2 s} &= \frac{(nk_2)!}{(nk_1)!(n(k_2 - k_1))!} 3^{-nk_2 s} \\ \approx & \frac{3^{-nk_2 s}(nk_2 e^{-1})^{nk_2}\sqrt{2\pi n k_2}}{(nk_1 e^{-1})^{nk_1} e^{-nk_1}\sqrt{2\pi n k_1}(n(k_2 - k_1)e^{-1})^{n(k_2-k_1)}\sqrt{2\pi n(k_2 - k_1)}} \\ = & \frac{3^{-nk_2 s}(k_2)^{nk_2}\sqrt{k_2}}{(k_1)^{nk_1}\sqrt{k_1}(k_2 - k_1)^{n(k_2-k_1)}\sqrt{2\pi n(k_2 - k_1)}}.\end{aligned}$$

Note, for any $c > 0$,

$$\lim_{n\to\infty} \sqrt[n]{c} = 1 = \lim_{n\to\infty} \sqrt[n]{n}.$$

Therefore,

$$\begin{aligned}&\lim_{n\to\infty} \sqrt[n]{\frac{3^{-nk_2 s}(k_2)^{nk_2}\sqrt{k_2}}{(k_1)^{nk_1}\sqrt{k_1}(k_2 - k_1)^{n(k_2-k_1)}\sqrt{2\pi n(k_2 - k_1)}}} \\ = \;& 3^{-k_2 s}\left(\frac{k_2^{k_2}}{k_1^{k_1}(k_2 - k_1)^{(k_2-k_1)}}\right).\end{aligned}$$

The abscissa of convergence is given by

$$\sigma = \frac{1}{k_2}\log_3\left(\frac{k_2^{k_2}}{k_1^{k_1}(k_2 - k_1)^{(k_2-k_1)}}\right),$$

when $k_1, k_2 \in \mathbb{N}$ and $k_1 < k_2$. When $k_1 = 0$ and $k_2 = 1$ or $k_1 = k_2 = 1$, the abscissa of convergence is zero, as above.

Using the above formula for the abscissa of convergence, σ may be considered as a function in the two variables k_1 and k_2. However, as with the regularity $\alpha(k_1, k_2)$, the value of σ depends only on the ratio $x = k_1/k_2$. Indeed, suppose $k_1 = xk_2$. Then

$$\begin{aligned}\sigma &= \log_3 k_2 - \left(\frac{k_1}{k_2}\right)\log_3 k_1 - \left(\frac{k_2 - k_1}{k_2}\right)\log_3(k_2 - k_1)\\ &= \log_3 k_2 - x\log_3(xk_2) - (1-x)\log_3((1-x)k_2)\\ &= -x\log_3 x - (1-x)\log_3(1-x).\end{aligned}$$

Temporarily treating σ as a function of one real variable x in the interval $(0, 1)$ allows one to show that σ attains it maximum at $x = 1/2$ and that its second derivative is strictly less than zero on $(0, 1)$. Indeed,

$$\begin{aligned}\sigma'(x) &= \log_3\left(\frac{1-x}{x}\right) \qquad \text{and}\\ \sigma''(x) &= \frac{1}{x(x-1)\ln 3}.\end{aligned}$$

On the interval $(0, 1)$, this function attains its maximum at $x = 1/2$ and is concave down. Further,

$$\begin{aligned}\lim_{x\to 0}\sigma(x) &= \lim_{x\to 1}\sigma(x) = 0,\\ \lim_{x\to 0^+}\sigma'(x) &= \infty, \qquad \text{and}\\ \lim_{x\to 1^-}\sigma'(x) &= -\infty\end{aligned}$$

This breakdown of the properties of the abscissa of convergence function σ also applies to the corresponding functions from Theorem 8.9 below.

When x is rational, there is a linear change of variable formula between x and the regularity values $\alpha(k_1, k_2)$. Specifically,

$$x = \frac{1-\alpha}{\log_3 2}.$$

Thus, σ is a function of the countable collection of regularity values which comprise a dense subset of the interval $[1 - \log_3 2, 1]$.

This function attains its maximum at $\alpha(1, 2) = 1 - (1/2)\log_3 2$, where σ is symmetric about this regularity value. The maximum value of σ coincides with Minkoski dimension of the support of μ, the Cantor Set. This may be a bit surprising because the lengths that constitute the partition zeta function differ significantly from those that constitute the Cantor String. Furthermore, the result holds in greater generality, see Theorem 8.9.

The structure of the graph of σ versus α is reminiscent of the multifractal spectrum in [8], see Figure 17 (which is best understood in the more general setting described below). A key difference is that the function σ is, again, defined only for a countable number of regularity values. In order to consider other values, the partition family $\mathfrak{P}$ must be enlarged or otherwise changed. Even then the measure μ may not yield enough structure with respect to these other regularity values to "complete" the graph of σ on the full interval $[1 - \log_3 2, 1]$.

The structure of the families of partition zeta functions for other measures which are supported on more general Cantor-like sets is readily accessible and follows directly from the computation in this section. A natural choice for the family of partitions $\mathfrak{P}$ is dictated by the construction of the measure itself. Indeed, for such measures a natural choice of $\mathfrak{P}$ is the disjoint union of the closed intervals in the construction at each stage along with the open intervals that comprise the gaps in between.

Example 8.6 (**Measures on Generalized Cantor Sets**).

Let us specify the context for the generalization mentioned above by reconsidering the construction in Figures 14 and 15. The set E_1 consists of two intervals of length 1/3, with mass split in ratios of 1/3 on the left and 2/3 to the right. Replace the two closed intervals of length 1/3 with two of length h^{-1} where h is a real number such that $h \geq 2$ (this guarantees that $2h^{-1} \leq 1$), keeping the left and right endpoints at 0 and 1, respectively. Replace the smaller mass 1/3 with w^{-1} where w is a real number such that $w \geq 2$ (so $w^{-1} + w^{-1}(w-1) = 1$). Iterating the construction process results in a (usually multifractal) measure $\mu(h, w)$ which is supported on a Cantor-like set.

Some special cases: $\mu(3, 2) = \rho$ and $\mu(3, 3) = \mu$ (both from above in this section); $\mu(2, 3) = \beta$, the binomial measure discussed below; and $\mu(2, 2) = \lambda$, the Lebesgue measure on the unit interval.

The measures $\mu(h, w)$ have a natural family of partitions arising from their construction.

Definition 8.7. *For a fixed $h > 2$, the <u>natural family of partitions</u> $\mathfrak{P}$ satisfies the following: At each stage $n \in \mathbb{N}$, the partition $\mathcal{P}_n$ consists of 2^n closed intervals with length h^{-1} and 2^{n-1} open intervals of length $(1 - 2h^{-1})h^{-(n-1)}$ along with the open intervals from the previous stages.*

The case $h = 2$ (where every interval has mass) is discussed in Proposition 8.8 below.

Redefining the lengths and the distribution of mass affects the pertinent regularity values $\alpha(k_1, k_2)$ and the function σ which gives the abscissa of convergence as a function of the regularity value. For a fixed pair $h \geq 2$ and $w \geq 2$, the regularity values arising from the construction of the measure $\mu(h,w)$ are given by

$$\begin{aligned} \alpha(k_1,k_2) &= \frac{\log\left(\frac{1}{w^{k_2-k_1}}\cdot\frac{(w-1)^{k_1}}{w^{k_1}}\right)}{\log\left(\frac{1}{h^{k_2}}\right)} \\ &= \log_h w - \frac{k_1}{k_2}\log_h(w-1), \end{aligned}$$

where k_1 and k_2 are necessarily relatively prime. The temporary variable $x = k_1/k_2$ now satisfies the relation

$$x = \frac{-(\alpha - \log_h w)}{\log_h(w-1)}.$$

The construction of the partition zeta functions for the measure μ generalizes to that of the measures $\mu(h,w)$, yielding the following proposition.

Proposition 8.8. *For a measure $\mu(h,w)$ with $h \geq 2$, $w > 2$, natural family of partitions $\mathfrak{P}$ and regularity $\alpha(k_1,k_2)$ where k_1 and k_2 are relatively prime non-negative integers such that $k_1 < k_2$, the partition zeta functions are of the form*

$$\zeta_{\mathfrak{P}}^{\mu(h,w)}(\alpha(k_1,k_2),s) = \sum_{n=1}^{\infty}\binom{nk_2}{nk_1}h^{-nk_2 s},$$

for Res *large enough.*

Proof. Case 1: $h > 2$.

This case follows straight from the above computation with measure $\mu = \mu(3,3)$. In particular, for regularity

$$\alpha(k_1,k_2) = \log_h w - \frac{k_1}{k_2}\log_h(w-1)$$

the binomial coefficients $\binom{nk_2}{nk_1}$ appear due to the fact that the construction of these measures uses two closed intervals with identical length h^{-1} ($1/3$ in the case of μ). Further, the restriction $w > 2$ ensures the appropriate breakdown of the regularity values (see Figure 16), specifically since the two closed subintervals in the first stage have different mass. Thus,

$$\zeta_{\mathfrak{P}}^{\mu(h,w)}(\alpha(k_1,k_2),s) = \sum_{n=1}^{\infty}\binom{nk_2}{nk_1}h^{-nk_2 s}.$$

Case 2: $h = 2$.

The formula in Case 1 is valid for $h = 2$ when the family $\mathfrak{P}$ is chosen as the dyadic intervals. For definiteness, choose the partitions which consist of half-open intervals with the left end closed, and the exception being the right-most interval, which is taken to be closed. Then

$$\zeta_{\mathfrak{P}}^{\mu(2,w)}(\alpha(k_1,k_2),s) = \sum_{n=1}^{\infty} \binom{nk_2}{nk_1} 2^{-nk_2 s}.$$

$\square$

Following the computation of the abscissa of convergence function σ in the case of the measure μ above leads to similar formulae for the measures $\mu(h,w)$ (which amounts to switching 3 with h). For practical purposes, σ is more easily computed as a function of the parameters k_1 and k_2, having the form

$$\sigma = \frac{1}{k_2} \log_h \left(\frac{k_2^{k_2}}{k_1^{k_1}(k_2-k_1)^{k_2-k_1}} \right).$$

This function attains a maximum value which is equal to the Minkowski dimension of the support of the corresponding measure $\mu(h,w)$.

Theorem 8.9. *As a function of the regularity values, the abscissa of convergence function σ for the measure $\mu(h,w)$ with $h \geq 2$ and $w > 2$ has the form*

$$\begin{aligned} \sigma(\alpha) &= \frac{(\alpha - \log_h w)}{\log_h (w-1)} \cdot \log_h \left(\frac{-(\alpha - \log_h w)}{\log_h (w-1)} \right) \\ &\quad - \left(1 + \frac{(\alpha - \log_h w)}{\log_h (w-1)}\right) \cdot \log_h \left(1 + \frac{(\alpha - \log_h w)}{\log_h (w-1)}\right). \end{aligned}$$

As the abscissa of convergence function, σ is defined on a dense subset of the interval $[\log_h w - \log_h (w-1), \log_h w]$, and it attains its maximum at

$$\alpha = \alpha(1,2) = \log_h w - (1/2)\log_h (w-1).$$

This maximum value coincides with the Minkowski dimension of the support of the measure $\mu(h,w)$. That is,

$$\begin{aligned} \dim_M(supp(\mu(h,w))) &= \max\{\sigma(\alpha) \mid \alpha = \alpha(k_1,k_2), (k_1,k_2) = 1\} \\ &= \log_h 2. \end{aligned}$$

Proof. For $h \geq 2$,

$$\begin{aligned} \sigma &= \frac{1}{k_2} \log_h \left(\frac{k_2^{k_2}}{k_1^{k_1}(k_2-k_1)^{k_2-k_1}} \right) \\ &= -x \log_3 x - (1-x) \log_3 (1-x). \end{aligned}$$

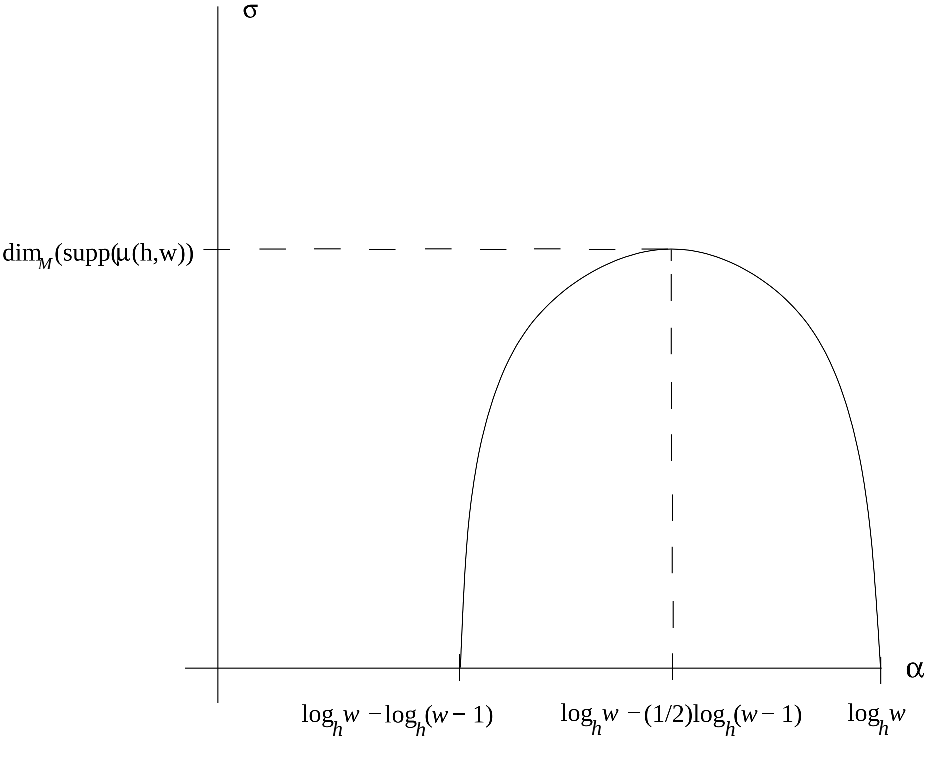

Figure 17: *σ as a function of α.*

where $x = k_1/k_2$. Using the relation

$$x = \frac{-(\alpha(k_1, k_2) - \log_h w)}{\log_h (w-1)}$$

gives the desired result for the form of σ. This function is concave down, symmetric about $\alpha = \alpha(1,2) = \log_h w - (1/2)\log_h (w-1)$, and has limit 0 at $\log_h w - \log_h (w-1)$ and $\log_h w$, following the construction in the case of $\mu = \mu(3,3)$.
Case 1: $h > 2$.

The maximum value of σ is attained when $k_1 = 1$ and $k_2 = 2$, which yields $\sigma(\alpha(1,2)) = \log_h 2$. This equality follows readily from

$$\sigma = \frac{1}{k_2} \log_h \left(\frac{k_2^{k_2}}{k_1^{k_1}(k_2 - k_1)^{k_2 - k_1}} \right).$$

To determine the Minkowski dimension of the support of the measure $\mu(h, w)$, compute the geometric zeta function of its complement. For every $n \in \mathbb{N}$, there are 2^{n-1} open intervals of length $(1 - 2h^{-1})h^{-(n-1)}$ in this fractal string. With these lengths denoted $\mathcal{L}$, the geometric zeta function is

$$\begin{aligned} \zeta_{\mathcal{L}}(s) &= \sum_{n=1}^{\infty} 2^{(n-1)}(1-2h^{-1})^s h^{-(n-1)s} \\ &= \frac{(1-2h^{-1})^s}{1-2h^{-s}}. \end{aligned}$$

The abscissa of convergence of this function is $\log_h 2$. See Figure 17.
Case 2: $h = 2$.

The maximum of the function σ is 1 in this case. The measures $\mu(2, w)$ are supported on the unit interval, which has Minkowski dimension 1. Note: The complement of the support in this case is the empty set, thus there is no corresponding geometric zeta function. □

Remark 8.10. The case of $\mu(h, 2)$ for $h \geq 2$ is simple. For the Lebesgue measure $\mu(2,2) = \lambda$ there is no regularity value aside from $\alpha = 1$ corresponding to any interval, regardless of the choice of $\mathfrak{P}$. For $\mu(h, 2)$ where $h > 2$, the behavior follows that of the measure $\mu(3,2) = \rho$, with the Minkowski dimension of the support recovered therein.

Remark 8.11. Figure 17 originally modeled the graph of the function σ associated with the measure $\mu = \mu(3,3)$. However, the shape is indicative of the behavior in general for $h \geq 2$ and $w > 2$, thus the graph has its current form. Note: The range of σ depends only on h, whereas the domain depends on both h and w. This corresponds to the fact that both h and w affect the regularity values α, but only the lengths of the intervals (powers of h^{-1}) from the construction of the support of the measures contribute to the partition zeta functions.

Figure 18: *Fourth stage in the construction of the binomial measure.*

Example 8.12 (The Binomial Measure).

The binomial measure β is a classic example of a multifractal measure. See Figure 18 for the fourth stage of the construction of β. This measure is supported on the unit interval and has multifractal structure in the sense that, for a continuum of pointwise (Hölder) regularity values, the support essentially breaks down into disjoint subsets that yield a corresponding the continuum of dimension values. The graph of this correspondence is called the multifractal spectrum of the measure β and has structure quite similar to the function σ.

In the setting of this text, $\beta = \mu(2, 3)$. In the construction of β, the masses $1/3$ and $2/3$ are placed on the left and right half-intervals, respectively, with successive half-intervals receiving the same mass distribution. A natural choice for the family of partitions $\mathfrak{P}$ in this case is the dyadic intervals. For definiteness, these dyadic intervals taken as in the proof of Proposition ??. Indeed, the proposition applies to this measure and yields the forms given below. Theorem 8.9 applies as well.

The fact that β is supported on the entire unit interval $[0, 1]$ means that no intervals in $\mathfrak{P}$ have regularity $\alpha = \infty$. The pertinent regularity values, again denoted $\alpha(k_1, k_2)$, are now given by

$$\alpha(k_1, k_2) = \frac{\log\left(2^{nk_1}/3^{nk_2}\right)}{\log\left(1/2^{nk_2}\right)} = \log_2 3 - \frac{k_1}{k_2},$$

where $k_1, k_2 \in \mathbb{N}$ and the pair (k_1, k_2) is necessarily relatively prime.

The forms of the partition zeta functions of the binomial measure β with these regularity values and the family $\mathfrak{P}$ of dyadic intervals are,

$$\zeta_{\mathfrak{P}}^{\beta}(\alpha(k_1, k_2), s) = \zeta_{\mathfrak{P}}^{\beta}(\alpha(k_2 - k_1, k_2), s) = \sum_{n=1}^{\infty} \binom{nk_2}{nk_1} 2^{-k_2 ns},$$

where the pair (k_1, k_2) is necessarily relatively prime as before.

The question of determining the domains of validity and abscissae of convergence is, on the surface, more daunting than its twin in the case of measure μ. Indeed, the sum of the lengths of the intervals which contribute to the partition zeta functions is infinity. Nevertheless, Proposition 8.8 and Theorem 8.9 indicate that there is structure to be had. The maximal abscissa of convergence value (the maximum of σ) for all the partition zeta functions associated to the measure β with the family $\mathfrak{P}$ is the one corresponding to regularity $\alpha(1, 2) = \log_2 3 - 1/2$. This maximum value is 1, which is of course the Minkowski dimension of the support of β (the unit interval). The symmetry and parabola-like shape of σ as a function of the countable collection of regularity values $\alpha(k_1, k_2)$ is also derived as above. See Figure 17.

The partition zeta functions provide a lot of promising multifractal information in the forms of Proposition 8.8 and Theorem 8.9, but this text leaves a lot to be desired. Generalizing the forms of the partition zeta functions to fit a larger class of measures, finding their meromorphic extensions and poles, and possibly finding an appropriate setting for the completion of the graph of a given abscissa of convergence function σ are all intriguing problems that arise from this section. Other suggestions for future research are provided in the next section.

9 Concluding Comments

The generalization of geometric zeta functions of fractal strings to multifractal zeta functions of measures provides new topological and geometric information for fractal strings as well as a new approach to multifractal analysis of measures. Investigating the properties of the family of multifractal zeta functions for a given measure has proven to be quite complicated and has spawned yet another type of zeta function called the partition zeta function. These zeta functions reflect properties of multifractal measures which are illuminated by current approaches to multifractal analysis.

This text provides many directions for future research. In the context of fractal strings, multifractal zeta functions may be applied to the new theory of complex dimenions of random zeta functions of random fractal strings as developed in [10]. Extending results further to random multifractal measures using partition zeta functions should prove to be an interesting and challenging problem.

The techniques used in the investigation of the measures of the form $\mu(h, w)$ from the previous section should extend readily to measures with even more general construction. For instance, the length of the intervals in the partitions may be allowed to vary with respect to both multiplicity and size. Each different approach to the construction of such measures presents its own difficulties for determining the family of partition zeta functions. Under certain conditions, if the multiplicities in the construction of the support increase, the number of intervals with the same regularity may be given by a multinomial coefficient. With $\sum_{j=1}^{N} k_j = q$, the multinomial coefficients are

$$\binom{q}{k_1 \ldots k_N} = \frac{q!}{k_1! \ldots k_n!}.$$

The partition zeta functions would then have forms similar to

$$\sum_{n=1}^{\infty} \binom{nq}{nk_1 ... nk_N} h^{-nqs}$$

for some $h \geq 2$. The breakdowns of the distribution of mass and the partitions will have interesting effects on the forms of the partition zeta functions.

Extending the multifractal and partition zeta functions to sets and measures of higher dimension is a good problem to consider. A useful guide in this endeavor, as mentioned in [20], should be provided by the recent work of Lapidus and Pearse on the complex dimensions of the Koch snowflake curve (see [22], as summarized in [29], §12.3.1). Also, more generally but from a different point of view, on the zeta functions and complex dimensions of self-similar fractals and tilings in $\mathbb{R}^d$ (see [23], as briefly described in [29], §12.3.2).

The properties of the complex dimensions with parameter α and the poles of the partition zeta functions deserve a more detailed investigation. Perhaps the structure of these poles contain information about multifractal measures akin to that of the usual complex dimensions (the poles of the geometric zeta functions) and fractal strings. Notions of localized zeta functions may also be created to investigate the local properties of fractal strings and multifractal measures.

Other types of zeta functions may stem from the construction of the multifractal and partition zeta functions, perhaps following other approaches of multifractal analysis, such as wavelet analysis. The work in [12, 13, 14, 15, 16] along with this text should provide a good foundation for such a venture.

These problems will likely be difficult both technically and conceptually. Hopefully the results of this paper will provide a decent foundation for further exploration in the fields of multifractal analysis and fractal geometry.

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