Partition Zeta Functions, Multifractal Spectra, (and Tapestries of Complex Dimensions)

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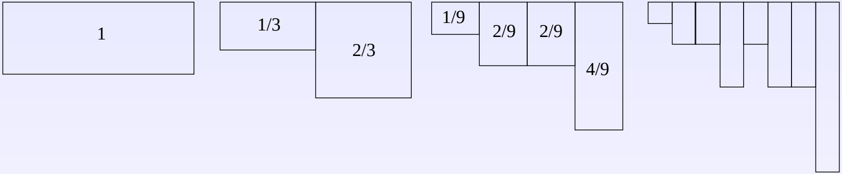
Weighted Iterated Functions System (WIFS)



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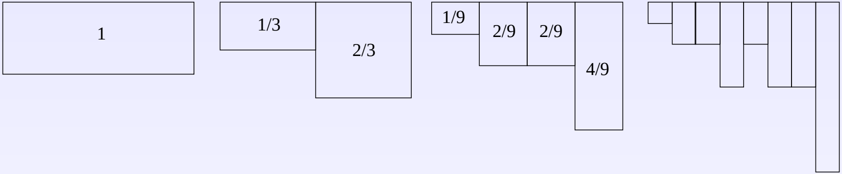

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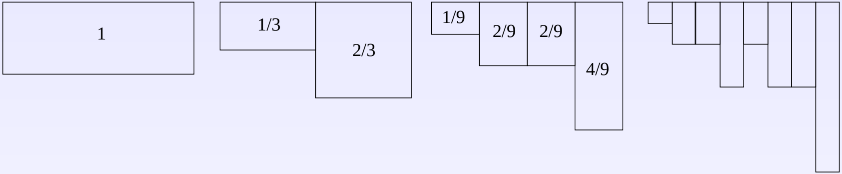

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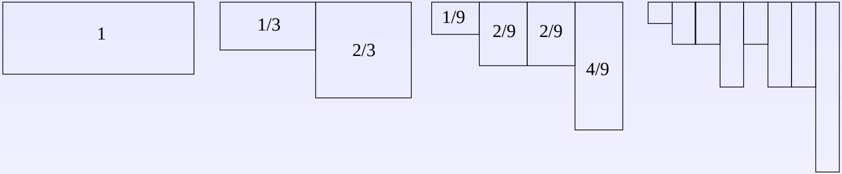

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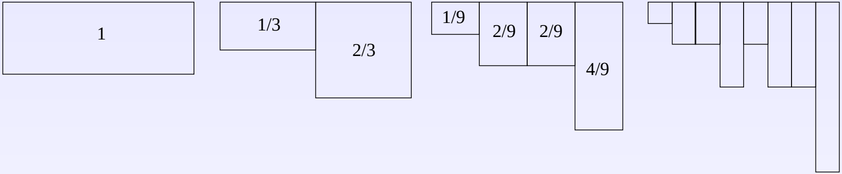

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Each WIFS defines a self-similar measure μ with support E uniquely defined as the nonempty compact set such that:

$$E = \bigcup_{j=1}^N S_j(E)$$

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Consider the measure $\mu(B(x,r))$ of the closed ball $B(x,r)$ with center x and radius r.

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This t is called *local Hölder regularity*.

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The *multifractal spectrum* of the measure μ is the function

$$f(t) = \dim_H(E_t).$$

Multifractal spectrum

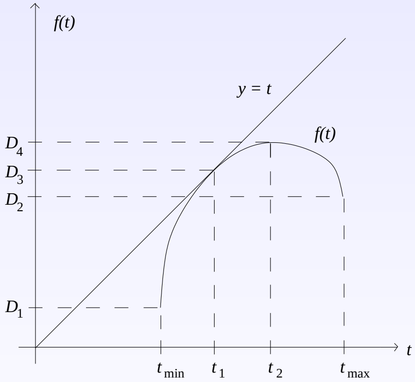

Figure: *The graph of the multifractal spectrum $f(t)$ of a self-similar measure μ where t is local Hölder regularity.*

Properties of the multifractal spectrum

- $t_{\min} = \min_j \left\{ \log_{r_j} p_j \mid j \in \{1, \ldots, N\} \right\}$.

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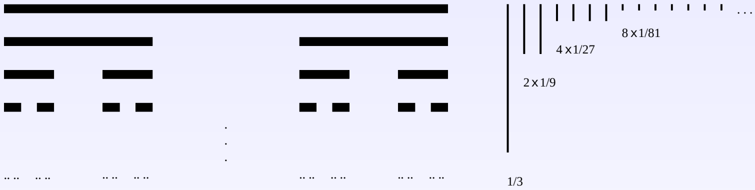
Fractal Strings



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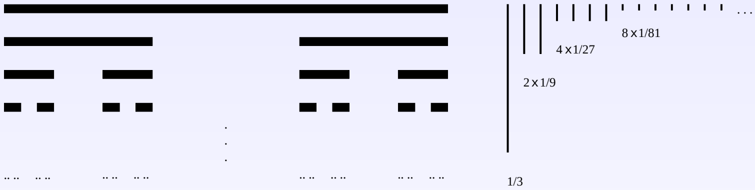

Definition

A *fractal string* Ω *is a bounded open subset of the real line. The collection of lengths* ℓ_j *of the disjoint intervals is denoted* $\mathcal{L}$.

Lengths and box dimension

Theorem

If a fractal string Ω in $[0,1]$ is of total length 1 and has an infinite number of lengths in its sequence $\mathcal{L}$, then

$$\dim_B(\partial\Omega) = \inf\left\{\sigma \in \mathbb{R} \;\middle|\; \sum_{j=1}^{\infty} \ell_j^{\sigma} < \infty\right\}.$$

The infimum on the right-hand side of the equation is the *abscissa of convergence* of the Dirichlet series $\sum_{j=1}^{\infty} \ell_j^{\sigma}$.

Geometric zeta functions and complex dimensions

Definition

The geometric zeta function of a fractal string Ω with lengths $\mathcal{L}$ is

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} \ell_j^s = \sum_{n=1}^{\infty} m_n l_n^s$$

where $\mathrm{Re}(s)$ *is large enough.*

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Definition

The set of (visible) complex dimensions of a fractal string Ω *with lengths* $\mathcal{L}$ *is*

$$\mathcal{D}_{\mathcal{L}}(W) := \{\omega \in W \mid \zeta_{\mathcal{L}} \text{ has a pole at } \omega\}.$$

Furthermore, if $W = \mathbb{C}$*, then* $\mathcal{D}_{\mathcal{L}} := \mathcal{D}_{\mathcal{L}}(\mathbb{C})$ *is simply called the set of complex dimensions of* $\mathcal{L}$*.*

Geometric counting functions

The *geometric counting function* of $\mathcal{L}$, denoted $N_{\mathcal{L}}(x)$, is defined by

$$N_{\mathcal{L}}(x) := \#\{j \geq 1 \mid \ell_j^{-1} \leq x\} = \sum_{n \geq 1 \mid l_n^{-1} \leq x} m_n.$$

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Proposition

Let Ω be a fractal string with lengths $\mathcal{L}$ such that $\mathcal{D}_{\mathcal{L}}(W)$ consists entirely of simple poles. Then, under certain mild growth conditions on $\zeta_{\mathcal{L}}$, we have

$$N_{\mathcal{L}}(x) = \sum_{\omega \in \mathcal{D}_{\mathcal{L}}(W)} \frac{x^{\omega}}{\omega} \operatorname{res}(\zeta_{\mathcal{L}}(s); \omega) + \{\zeta_{\mathcal{L}}(0)\} + R(x),$$

where $R(x)$ is an error term of small order and the term in braces is included only if $0 \in W \backslash \mathcal{D}_{\mathcal{L}}(W)$.

For the Cantor string

The geometric zeta function of the Cantor string is

$$\zeta_{CS}(s) = \sum_{n=1}^{\infty} 2^{n-1} 3^{-ns} = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}}.$$

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The geometric counting function of the Cantor string is

$$N_{CS}(x) = \frac{1}{2 \log 3} \sum_{m=-\infty}^{\infty} \frac{x^{D+imp}}{D + imp} - 1,$$

where $D = \log_3 2$ is the Minkowski dimension of the Cantor string (technically of the Cantor set) and $p = 2\pi / \log 3$ is its oscillatory period.

Regularity

Definition

The <u>regularity</u> $A(U)$ of an interval $U \subset [0,1]$ with respect to a (Borel) measure μ is

$$A(U) = \frac{\log \mu(U)}{\log |U|},$$

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Equivalently, $A(U)$ is the exponent α that satisfies

$$|U|^{\alpha} = \mu(U).$$

α-Lengths of μ

Definition

The α-lengths of a measure μ with respect to a sequence of partitions $\mathfrak{P} = \{\mathcal{P}_n\}_{n=1}^{\infty}$ with mesh tending to zero is given by

$$\mathcal{L}_{\mathfrak{P}}^{\mu}(\alpha) := \left\{ \ell | \ell = |P_n^k| \text{ and } A(P_n^k) = \alpha \text{ for some } P_n^k \in \mathcal{P}_n \in \mathfrak{P} \right\}.$$

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Attained regularity values

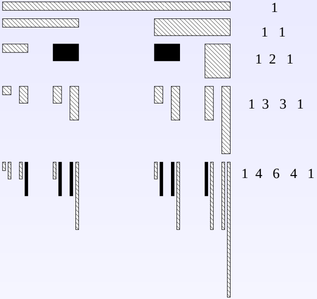

Figure: *The regularity value* $\alpha = 1 - (1/2)\log_3 2$ *is attained at every partition* $\mathcal{P}_K$ *where* K *is an even positive integer.*

The intervals P with positive mass in the partitions $\mathcal{P}_K$ satisfy $|P| = r_1^{k_1} \cdots r_N^{k_N}$ and $\mu(P) = p_1^{k_1} \cdots p_N^{k_N}$ where $\mathbf{k} = (k_1, \ldots, k_N) \in (\mathbb{N} \cup \{0\})^N$ such that $\sum_{j=1}^N k_j = K$.

Attained regularity values

Consider $\mathbf{k}$ where $\gcd(k_1, \ldots, k_N) = 1$ and $\mathbf{k} \neq \mathbf{0}$. Define $\alpha(\mathbf{k})$ by

$$\alpha(\mathbf{k}) := \frac{\log p_1^{k_1} \cdots p_N^{k_N}}{\log r_1^{k_1} \cdots r_N^{k_N}}.$$

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Thus, every interval $P \in \mathcal{P}_K \in \mathfrak{P}$ such that $|P| = r_1^{nk_1} \cdots r_N^{nk_N}$ and $\mu(P) = p_1^{nk_1} \cdots p_N^{nk_N}$ for some $n \in \mathbb{N}$ satisfies

$$A(P) = \alpha(\mathbf{k}).$$

Partition ZFs

Definition

For a measure μ and ordered family of partitions $\mathfrak{P} = \{\mathcal{P}_n\}_{n=1}^{\infty}$ which mesh to zero, the partition zeta function with regularity α is

$$\zeta_{\mathfrak{P}}^{\mu}(\alpha, s) = \zeta_{\mathcal{L}_{\mathfrak{P}}^{\mu}(\alpha)}(s) = \sum_{n=1}^{\infty} \sum_{A(P_n^k)=\alpha} |P_n^k|^s,$$

where the P_n^k are the disjoint intervals in the partitions $\mathcal{P}_n$ (and $\mathrm{Re}(s)$ *is large enough).*

Regularity breakdown

Lemma

With μ *and* $\mathfrak{P}$ *as above, and with* $\mathbf{p}' = (p_1', \ldots, p_w')$ *as the* w *distinct values among the* N *components of* $\mathbf{p}$*, if* $\gcd(k_1', \ldots, k_w') = 1$ *and the numbers* $\log_r p_1', \ldots, \log_r p_w'$ *are rationally independent, then the distinct regularity values attained by* μ *on* $\mathfrak{P}$ *are given by*

$$\alpha(k_1', \ldots, k_w') = \frac{1}{K} \log_r \left((p_1')^{k_1'} \ldots (p_w')^{k_w'} \right).$$

Moreover, for every $n \in \mathbb{N}^*$*, the number of intervals* P *with regularity value*

$$\alpha(nk_1', \ldots, nk_w') = \alpha(k_1', \ldots, k_w')$$

in the partition $\mathcal{P}_{nK}$ *is*

$$\binom{nK}{nk_1, \ldots, nk_N} = \binom{nK}{nk_1', \ldots, nk_w'} c_1^{nk_1'} \ldots c_w^{nk_w'}.$$

Form of the partition zeta functions

Proposition

If the conditions of the Lemma are satisfied, then

$$\zeta^{\mu}_{\mathfrak{P}}(\alpha(\mathbf{k}), s) = \sum_{n=1}^{\infty} \binom{nK}{nk'_1, \dots, nk'_w} c_1^{nk'_1} \cdots c_w^{nk'_w} r^{nKs}$$

and

$$f^{\mu}_{\mathfrak{P}}(\alpha(\mathbf{k})) = f^{\mu}_{\mathfrak{P}}(\alpha(k_1, ..., k_N)) = \log_{r^K} \left(\frac{(k'_1)^{k'_1} \cdots (k'_w)^{k'_w}}{c_1^{k'_1} \cdots c_w^{k'_w} K^K} \right).$$

Part one of a first result

Theorem

Assume the conditions of the Lemma are satisified and that there are $w = 2$ distinct values among $p_1, \ldots, p_N$, and $r_1 = \ldots = r_N = r$. Then

$$\zeta_{\mathfrak{P}}^{\mu}(\alpha(\mathbf{k}), s) = \zeta_{\mathfrak{P}}^{\mu}(\alpha(k_2', K), s) = \sum_{n=1}^{\infty} \binom{nK}{nk_2'} (N - c_2)^{n(K-k_2')} c_2^{nk_2'} r^{nKs}$$

and

$$f_{\mathfrak{P}}^{\mu}(\alpha(\mathbf{k})) = f_{\mathfrak{P}}^{\mu}(\alpha(k_2', K)) = \log_{r^K} \left(\frac{(K - k_2')^{K-k_2'} (k_2')^{k_2'}}{(N - c_2)^{K-k_2'} c_2^{k_2'} K^K} \right).$$

(c_1 and c_2 denote the multiplicities of the two distinct values among $p_1, \ldots, p_N$.)

Part two of a first result

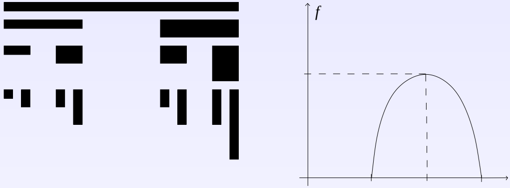
Theorem

Moreover, the concave envelope $\hat{f}_{\mathfrak{P}}^{\mu}$ of $f_{\mathfrak{P}}^{\mu}$ has infinite slope at the extreme values of the attained regularity values.

Lastly, we have

$$\max_{\alpha(\mathbf{k})}\{f_{\mathfrak{P}}^{\mu}(\alpha(\mathbf{k}))\} = f_{\mathfrak{P}}^{\mu}(\alpha(c_2/N)) = \dim_H(\mathrm{supp}(\mu)).$$

PZFs of the binomial measure on the Cantor set



$$\alpha = \alpha(k_2', K) \quad = \quad \frac{\log\left(2^{nk_2'}/3^{nK}\right)}{\log\left(1/3^{nK}\right)} = 1 - \frac{k_2'}{K} \log_3 2,$$

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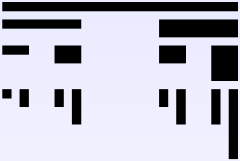

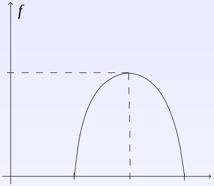

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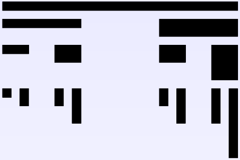

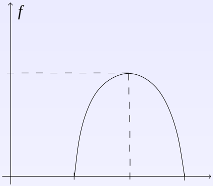

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$$\dim_B(supp(\nu)) = \max\{f(\alpha(k_2', K))\} = \log_3 2.$$

Multifractal spectrum of the binomial measure

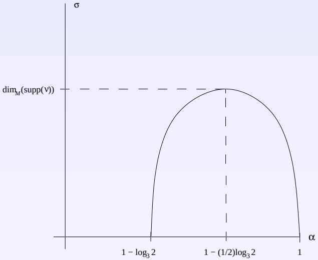
Corollary

Consider the binomial measure β_0 *defined by the similarities* $S_1(x) = x/2$ *and* $S_2(x) = x/2 + 1/2$ *with scaling ratios* $\mathbf{r} = (1/2, 1/2)$ *and the probability vector* $\mathbf{p} = (1/3, 2/3)$*. Then, for all* $t \in [t_{\min}, t_{\max}]$*, we have*

$$\hat{f}_{\mathfrak{P}}^{\beta_0}(t) = f_g(t) = f_s(t) = b^*(t),$$

where $\hat{f}_{\mathfrak{P}}^{\beta_0}$ *is the concave envelope of* $f_{\mathfrak{P}}^{\beta_0}$ *on the interval* $[t_{\min}, t_{\max}]$*. (Here,* $[t_{\min}, t_{\max}] = [\log_2 3 - 1, \log_2 3]$*.)*

Explicit form of the multifractal spectrum



The abscissa of convergence function $f(\alpha)$ for the binomial measure on the Cantor set is

$$\begin{aligned} f(\alpha) &= \frac{(\alpha-1)}{\log_3 2} \cdot \log_3\left(\frac{-(\alpha-1)}{\log_3 2}\right) \\ &\quad - \left(1 + \frac{(\alpha-1)}{\log_3 2}\right) \cdot \log_3\left(1 + \frac{(\alpha-1)}{\log_3 2}\right). \end{aligned}$$

Besicovitch subsets of a self-similar set

Definition

Let E be the self-similar set uniquely determined by an IFS. The ***Besicovitch subset*** *$E(\mathbf{q})$ of the set E is defined as follows:*

$$E(\mathbf{q}) := \left\{ x \in E \mid \lim_{k\to\infty} \frac{\#_j(x|_k)}{k} = q_j, j \in \{1, \ldots, N\} \right\},$$

where $x|_k$ is the truncation of x at its k-th digit in the expansion implied by the IFS and $\#_j(x|_k)$ is the number of times the digit j appears in $x|_k$.

Classic and recent results

(1934) Besicovitch: Binary expansion for $x \in [0,1]$:

$$\dim_H(E(\mathbf{q})) = \frac{-q_1 \log q_1 - q_2 \log q_2}{\log 2}.$$

(Recovered when $\mathbf{r} = (1/2, 1/2)$.)

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$$\dim_H(E(\mathbf{q})) = \frac{-q_1 \log q_1 - q_2 \log q_2}{\log 2}.$$

(Recovered when $\mathbf{r} = (1/2, 1/2)$.)

(1949) Eggleston: N-ary expansion for $x \in [0,1]$:

$$\dim_H(E(\mathbf{q})) = \frac{-\sum_{j=1}^{N} q_j \log q_j}{\log N}.$$

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(1992) Cawley and Mauldin: Expansion for $x \in E$ determined by an IFS:

$$\dim_H(E(\mathbf{q})) = \frac{\sum_{j=1}^{N} q_j \log q_j}{\sum_{j=1}^{N} q_j \log r_j}.$$

Part one of another result

Theorem

Consider an IFS that satisfies the OSC and yields μ *and* $\mathfrak{P}$ *with* $\mathbf{r}$ *and* $\mathbf{p}$*. Consider the following hypothesis:*

(H) *For all* $\mathbf{k} = (k_1, \ldots, k_N) \in (\mathbb{N} \cup \{0\})^N$ *where* $\gcd(k_1, \ldots, k_N) = 1$ *and* $\mathbf{k} \neq \mathbf{0}$*, the regularity values* $\alpha(\mathbf{k})$ *are distinct. That is, suppose* $\alpha(z_1, ..., z_N) = \alpha(\mathbf{k})$ *if and only if there exists* $m \in \mathbb{N}$ *such that* $z_j = mk_j$ *for all* $j \in \{1, \ldots, N\}$*.*

If (**H**) *holds with* $\gcd(k_1, \ldots, k_N) = 1$*, then letting* $K := \sum_{j=1}^{N} k_j$ *we have*

$$\zeta_{\mathfrak{P}}^{\mu}(\alpha(\mathbf{k}), s) = \sum_{n=1}^{\infty} \binom{nK}{nk_1 \cdots nk_N} (r_1^{k_1} \cdots r_N^{k_N})^{ns}.$$

Part two of another result

Theorem

Moreover, the abscissa of convergence $\sigma = f_{\mathfrak{P}}^{\mu}(\alpha(\mathbf{k}))$ *of the partition zeta function* $\zeta_{\mathfrak{P}}^{\mu}(\alpha(\mathbf{k}), s)$ *is given by*

$$f_{\mathfrak{P}}^{\mu}(\alpha(\mathbf{k})) = \frac{\sum_{j=1}^{N}(k_j/K)\log(k_j/K)}{\sum_{j=1}^{N}(k_j/K)\log r_j} = \dim_H(E(\mathbf{k}/K)),$$

where $E(\mathbf{k}/K)$ *is the Besicovitch subset of the self-similar fractal* $E = \mathrm{supp}(\mu)$. *Equivalently, and with use of the convention* $0^0 = 1$, *the abscissa of convergence* σ *is the unique real number satisfying the equation*

$$(r_1^{k_1} \cdots r_N^{k_N})^{\sigma} \frac{K^K}{k_1^{k_1} \cdots k_N^{k_N}} = 1;$$

in addition, $\sigma > 0$.

Conjecture

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For a self-similar measure μ on $[0,1]$ and its natural sequence of partitions $\mathfrak{P}$, we have

$$\hat{f}^{\mu}_{\mathfrak{P}}(t) = f_g(t) = f_s(t) = b^*(t),$$

for all $t \in [t_{\min}, t_{\max}]$, where $\hat{f}^{\mu}_{\mathfrak{P}}(t)$ is the concave envelope of $f^{\mu}_{\mathfrak{P}}(\alpha)$ on $[t_{\min}, t_{\max}]$.

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