

The Hindu-Arabic System

To gain a better understanding of our numeration system (also called the base ten system), we'll analyze how to represent numbers with other bases. We've already done this in some sense when we looked at the Babylonian (base 60) and Mayan systems (base 20 with 18 multiplier). Now we'll look at how to convert back and forth between other bases.

For our first example, let's pick base five. Below is what is called a chip abacus. Reading from right to left, each card represents increasing powers of 5, labelled $5^0, 5^1, 5^2, 5^3, \dots$. The cards indicate place value positions, and the number of chips on one card gives the "number of the specific position." The following picture represents the base ten numeral 147 in base five.

$$(1 * 5^3) + (0 * 5^2) + (4 * 5^1) + (2 * 5^0) = (1 * 125) + (4 * 5) + (2 * 1) = 125 + 20 + 2 = 147$$

If we think of the number of chips on each card as a "digit" we may rewrite 147 in base 5.

$$147 = (1 * 5^3) + (0 * 5^2) + (4 * 5^1) + (2 * 5^0) = 1042_{\text{five}}$$

We call 1042_{five} the **base five representation** of 147, and we read it as "one zero four two, base five." Notice 1042_{five} is not read as "one thousand forty-two, base five" because that type of description is reserved for base ten numbers.

1. (a) Draw a chip abacus diagram to represent 234 in base 5. Write out an equation similar to the one above to show your picture is correct.

- (b) Use your chip abacus to write the base five representation of 234.

The above examples show pictorially how to convert numbers from base ten to base five. Now let's go the other way. Let's convert 213_{five} to base ten.

$$213_{\text{five}} = (2 * 5^2) + (1 * 5^1) + (3 * 5^0) = (2 * 25) + (1 * 5) + (3 * 1) = 50 + 5 + 3 = 58$$

Typically we do not write 133_{ten} because base ten is our default base.

2. Write the base ten representation of the following.

(a) 413_{five}

(b) 1000_{five}

(c) 2020_{five}

When using base five, each card in the chip abacus contains no more than four chips ($5 - 1 = 4$). This is because if a card contains five chips, it can be exchanged for one chip in the next card.

Now let's work with different bases. If we create a new chip abacus for a different base, each card's capacity is (base number)-1 chips.

Examples:

(a) If we work in base three, each card in the abacus needs enough space for two chips.

(b) If we work in base four, each card in the abacus needs enough space for ___ chips. Why? Draw cards representing base four.

3. On each of the chip abaci from the previous page, represent 30 in base three and base four respectively.
4. Determine the base ten representation of each of the following.

(a) $1025_{\text{six}} =$

(b) $123_{\text{four}} =$

(c) $23_{\text{nine}} =$

What about bases larger than ten? As an example, let's look at base 12. Since $12 > 10$, we need extra digits in base twelve to represent the base ten numbers 10 and 11. We use T to represent 10, and E to represent 11. The digits in the base twelve system are

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T, E$$

This means $10_{\text{twelve}} = 12$.

5. How many chips would each card of a base twelve abacus need to hold? ____
6. Determine the base ten representation of each of the following.

(a) 123_{twelve}

(b) TE_{twelve}

Conversions: There is another way to convert from base ten to other bases, which we used in our discussion of ancient numeration systems. It involves long division! Our original example showed $147 = 1042_{\text{five}}$. We can apply long division to achieve the same result.

$$\begin{array}{r}
 1 \\
 125 \overline{)147} \\
 \underline{125} \\
 22
 \end{array}
 \qquad
 \begin{array}{r}
 0 \\
 25 \overline{)22} \\
 \underline{0} \\
 22
 \end{array}
 \qquad
 \begin{array}{r}
 4 \\
 5 \overline{)22} \\
 \underline{20} \\
 2
 \end{array}
 \qquad
 \begin{array}{r}
 2 \\
 1 \overline{)2} \\
 \underline{2} \\
 0
 \end{array}$$

Notice the quotients are the digits in the base five representation of 147.

Examples: Now you try:

7. For each base ten number given, find its representation in the specified base by using long division.
 - (a) 362 into base five
 - (b) 342 into base six
 - (c) 1584 into base twelve
 - (d) 28 into base three
 - (e) 11 into base two